Geometry of the trilogarithm and the motivic Lie algebra of a field

A.B. Goncharov

Contents

Summary. We express explicitly the Aomoto trilogarithm by classical trilogarithms and investigate the algebraic-geometric structures staying behind: different realizations of the weight three motivic complexes. Applying these results we describe the motivic structure of the Grassmannian tetralogarithm function and determine the structure of the motivic Lie coalgebra in degrees ≤ 4. Using this we give an explicit construction of the Borel regulator map

$$r_4 : K_7(\mathbb{C}) \longrightarrow \mathbb{R}$$

which together with the Borel theorem leads to results about $\zeta_F(4)$.

1 Introduction

The classical n-logarithm is defined by induction as an integral

$$Li_n(z) := \int_0^z Li_{n-1}(t) d\log t, \quad Li_1(z) = -\log(1-z)$$

So it can be written as an n-dimensional integral

$$Li_n(z) = \int_{0 \leq 1-t_1 \leq t_2 \leq \ldots \leq t_n \leq z} \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$$

Aomoto considered [A] more general integrals where the differential form $\frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n}$ is integrated over an arbitrary n-dimensional real simplex in $\mathbb{C}^n$. Let me recall this construction in a more formal setting.

Let F be a field. An n-simplex in $\mathbb{P}^n(F)$ is a collection of hyperplanes $L = (L_0, \ldots, L_n)$. It is nondegenerate if the intersection of the hyperplanes L_i is empty. A face of a simplex is any nonempty intersection of hyperplanes from L. A pair of simplices is *admissible* if L and M have no common faces of the same dimension. Now let $F = \mathbb{C}$. Then there is a canonical n-form ω_L in $\mathbb{CP}^{\ltimes}$ with logarithmic poles on the hyperplanes L_i. If $z_i = 0$ are homogeneous equations of L_i then

$$\omega_L = d\log(z_1/z_0) \wedge \ldots \wedge d\log(z_n/z_0)$$

Let Δ_M be an n-cycle representing a generator of $H_n(\mathbb{CP}^n, M)$.

The Aomoto n-logarithm is a multivalued function on configurations of admissible pairs of simplices $(L; M)$ in $\mathbb{C}P^n$ defined as follows:

$$\Lambda_n(L; M) := \int_{\Delta_M} \omega_L$$

The classical n-logarithm corresponds to a very special pair of simplices in $\mathbb{P}^n$, see fig.1, so it is a very special case of the Aomoto n-logarithm.

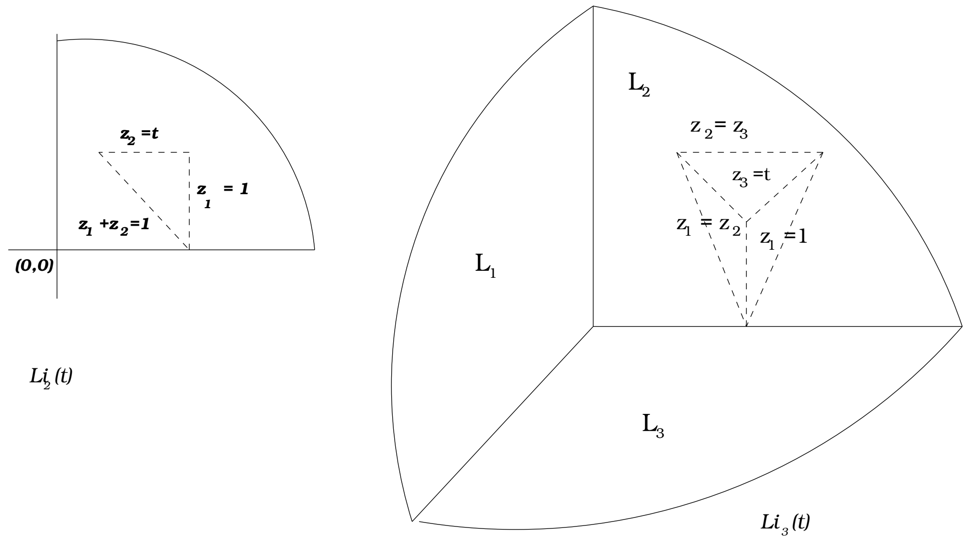

figure 1

In the present paper, which is a continuation of [BGSV1-2] and [G0-3], we express explicitly the Aomoto trilogarithm by classical trilogarithms and investigate the algebraic-geometric structures staying behind: different realizations of the weight three motivic complexes. For the Aomoto dilogarithm a similar problem was solved in [BGSV1-2].

The function $\Lambda_3(L; M)$ is defined on configurations, i.e. projective equivalence classes of $4+4$ points in $\mathbb{P}^3$ (vertices of a pair of tetrahedra), while the classical trilogarithm lives on $\mathbb{P}^1$. To build a bridge between these functions we relate each of them with the Grassmannian trilogarithm $\mathcal{P}_3^G$ defined on configurations of six points in $\mathbb{P}^2$ (see section 4.1 for a definition of Grassmannian polylogarithms). Our main geometric construction, the map a_3 defined in s. 3.3, (see its different versions on fig. 3, 8 and 9), permits to go from configurations of $4+4$ points in $\mathbb{P}^3$ to configurations of 6 points in $\mathbb{P}^2$. Then we apply the generalized cross-ratio map r_3 from [G0-3] to get to $\mathbb{P}^1$. The map a_3 sheds a new light on the key ansatz from [G0-3] leading to the functional equation for the classical trilogarithm, see fig. 10 and the discussion there.

The Aomoto n-logarithm for $n > 3$ **can not** be expressed by the classical n-logarithm. However the *explicit* relation between the Aomoto n-logarithm, which is defined on configurations of $2(n+1)$ points in $\mathbb{P}^n$, and the Grassmannian n-logarithm, which lives on configurations of $2n$ points in $\mathbb{P}^{n-1}$, should exist for all n. In section 4 we explain how such a relation would give an explicit construction of a certain graded co-Lie algebra $G_\bullet(F)$ over $\mathbb{Q}$, which should be isomorphic to the Lie coalgebra of the Galois group of the category of mixed

Tate motives over a field F. The cohomology of this Lie algebra should give the appropriate pieces of Quillen's K-theory of the field F modulo torsion. It would be very interesting to relate our approach with the work of M. Hanamura and R. MacPherson [HM].

There are several other candidates for the motivic Lie coalgebra, see [BK], [BMS-BGSV], [G5]. However all of them are constructed as Hopf algebras, so we get the Lie coalgebras as the quotient by the decomposable elements. Our approach should lead directly to a Lie coalgebra. The degree 2 and 3 parts of its standard cochain complex are precisely the Bloch-Suslin complex and the weight three motivic complex defined in [G0-3], so $G_\bullet(F)$ should be the smallest possible realization of the motivic Lie algebra.

In section 5 we define the structure of the motivic Lie coalgebra in degree 4, i.e. we define a cobracket

$$G_4(F) \longrightarrow G_3(F) \otimes G_1(F) \oplus \Lambda^2 G_2(F)$$

which satisfies the condition $\delta^2 = 0$ in $G_2(F) \otimes \Lambda^2 G_1(F)$. This together with the previous results of the author provide a description of the Lie subcoalgebra $G(F)_{\leq 4}$.

An immediate application of this is a description of the "fine" (or motivic) structure of the Grassmannian tetralogarithm function. In particular we get an explicit construction of the Borel regulator map

$$r_4 : K_7(\mathbb{C}) \longrightarrow \mathbb{R}$$

This together with the famous theorem of Borel leads to results about special values of the Dedekind zeta function at $s = 4$ (these results, however, are not sufficient to establish Zagier's conjecture at $s = 4$).

In the forthcoming paper [G8] we will complete this story by giving an explicit description of the general Beilinson regulator map in weight 4.

Our results partially generalize the work [BGSV] from $\mathbb{P}^2$ to $\mathbb{P}^3$. We discuss in s. 4 what remains to be done. An application to an explicit construction of the weight four motivic complexes will be discussed elsewhere.

I am extremely grateful to Herbert Gangl who wrote a proof of lemma (3.7), helped me to check coefficients in the formulas and pointed out a lot of errors in a preliminary version of the paper. Finally, I am very much indebted to the referee who spot a lot of misprints and made many useful remarks.

The results of this paper were obtained in May 1992 during my stay in the Max-Planck-Institute (Bonn). The paper was written in MPI later on (when I learned how to draw pictures using computer). I am very grateful to the MPI for hospitality and support.

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2 The scissors congruence groups of pairs of simplices in $\mathbb{P}^n(F)$

1.The scissors congruence groups $A_n(F)$. Let me recall some definitions from [BMS], [BGSV]. It is handier to work with configurations of points than with hyperplanes. Let us apply the projective duality $\mathbb{P}^n \longrightarrow \hat{\mathbb{P}}^n$ which transforms the configuration of $2(n+1)$ hyperplanes $(L_0, ..., L_n; M_0, ..., M_n)$ to a configuration of points $(l_0, ..., l_n; m_0, ..., m_n)$. Abusing notations we will denote it also $(L; M)$, where now $L = (l_0, ..., l_n)$ and $M = (m_0, ..., m_n)$.

The group $A_n(F)$ is generated by configurations of $2(n+1)$ points $(l_0, ..., l_n; m_0, ..., m_n)$ in $\mathbb{P}^n(F)$ which are vertices of admissible pairs of simplices subject to the following relations:

1)*Nondegeneracy.* $(L; M) = 0$ if $(l_0, ..., l_n)$ or $(m_0, ..., m_n)$ belong to a hyperplane.

2) *Skew symmetry.* $(\sigma L; M) = (L; \sigma M) = (-1)^{|\sigma|}(L; M)$ for any permutation $\sigma \in S_{n+1}$.

3)*Additivity.* For any configuration $(l_0, ..., l_{n+1})$

$$\sum_{i=0}^{n+1}(-1)^i(l_0, ..., \hat{l}_i, ..., l_{n+1}; m_0, ..., m_n) = 0$$

if all the terms are admissible (additivity in L). A similar condition is imposed for $(m_0, ..., m_{n+1})$ (additivity in M).

$\hat{3}$)*Dual additivity.* For any configuration $(l_0, ..., l_{n+1})$

$$\sum_{i=0}^{n+1}(-1)^i(l_i|l_0, ..., \hat{l}_i, ..., l_{n+1}; m_0, ..., m_n) = 0$$

if all the terms are admissible, as well as the similar condition is imposed for $(m_0, ..., m_{n+1})$. Here $(l|m_1, ..., m_n)$ denotes the configuration of n points in $\mathbb{P}^{n-1}$ obtained by the projection of the points m_i with the center at the point l.

4) *Projective invariance.* $(gL; gM) = (L; M)$ for any $g \in PGL_{n+1}(F)$.

These relations reflect properties of Aomoto polylogarithms.

The cross-ratio provides a canonical isomorphism

$$a_1 : A_1(F) \longrightarrow F^*, \quad a_1 : (l_0, l_1; m_0, m_1) \longmapsto r(l_0, l_1, m_0, m_1)$$

2. A coproduct on the generic part of $A_n(F)$ ([BMS], [BGSV]). Set $A_0 = \mathbb{Z}$. Let $A^0_n(F) \subset A_n(F)$ be the subgroup generated by pairs of simplices in generic position. Let us define a coproduct $\nu : A^0_n \longrightarrow \oplus_k A^0_k \otimes A^0_{n-k}$. Set for $k, n-k > 0$

$$\nu = \oplus \nu_{n-k,k}, \qquad \nu_{n-k,k} : A^0_n :\longrightarrow A^0_{n-k} \otimes A^0_k,$$

$$\nu_{n-k,k} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$
$$\sum_{I,J} (-1)^{\sigma(I,J)} (l_{i_1} ... l_{i_k} | l_0, ..., \hat{l}_{i_1}, ..., \hat{l}_{i_k}, ..., l_n; m_0, m_{j_1}, ..., m_{j_{n-k}})$$
$$\otimes (m_{j_1} ... m_{j_{n-k}} | l_0, l_{i_1}, ..., l_{i_k}; m_0, ..., \hat{m}_{j_1}, ..., \hat{m}_{j_{n-k}}, ..., m_n)$$

where $I := \{0 < i_1 < ... < i_k\}$, $J := \{0 < j_1 < ... < j_{n-k}\}$. Here $\sigma(I, J) = sign(I, \bar{I}) \cdot sign(J, \bar{J})$, where $sign(I, \bar{I})$ is the sign of the permutation $(1, ..., n) \to (I, \bar{I})$ (similarly for $sign(J, \bar{J})$). Here $\bar{I}$ is the complement to the set I in $\{0, ..., n\}$. For example

$$\nu_{2,1} : (l_0, ..., l_3; m_0, ..., m_3) \longmapsto$$
$$(l_3 | l_0, l_1, l_2; m_0, m_2, m_3) \otimes (m_2, m_3 | l_0, l_3; m_0, m_1) + ...$$

Proposition 2.1 *$(A_\bullet(F)^0, \nu)$ is a graded coalgebra.*

I will need this statement only in the case of degree 3. In this case it is not hard to deduce it from proposition (2.3) below.

Proposition 2.2 *$P : (L; M) \longrightarrow (M; L)$ defines an antiautomorphism of the graded coalgebra $(A_\bullet(F)^0, \nu)$.*

Proof follows from the definitions.

3. Another formula for $\nu_{1,n-1}$ and $\nu_{n-1,1}$. Unfortunately in the definition of the coproduct ν we have to choose vertices l_0 in L and m_0 in M first, so the skew-symmetry is not obvious. In the next proposition we give another formula for $\nu_{1,n-1}$ and $\nu_{n-1,1}$ which is skew-symmetric from the beginning and much more convenient.

Let V_n be an n-dimensional vector space over a field F. Choose a volume form $\omega_n \in det V_n^*$. For any n vectors $l_1, ..., l_n$ in V_n set

$$\Delta(l_1, ..., l_n) := < l_1 \wedge ... \wedge l_n, \omega_n >$$

Proposition 2.3 *(See figure 2)*

$$\nu_{1,n-1} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$
$$-\sum_{i,j=0}^{n} (-1)^{i+j} \Delta(m_j, l_0, ..., \hat{l}_i, ..., l_n) \otimes (m_j | l_0, ..., \hat{l}_i, ..., l_n; m_0, ..., \hat{m}_j, ..., m_n)$$
$$\nu_{n-1,1} : (l_0, ..., l_n; m_0, ..., m_n) \longmapsto$$
$$-\sum_{i,j=0}^{n} (-1)^{i+j} (l_i | l_0, ..., \hat{l}_i, ..., l_n; m_0, ..., \hat{m}_j, ..., m_n) \otimes \Delta(l_i, m_0, ..., \hat{m}_j, ..., m_n)$$

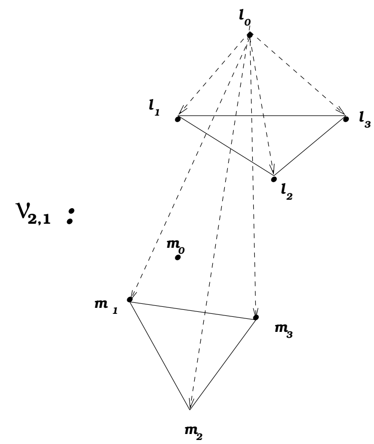


figure 2

Proof. Let us compute

$$\nu_{n-1,1}(l_0,...,l_n;m_0,...,m_n) = \tag{1}$$

$$\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes r(m_1...\hat{m}_j...m_n|l_0,l_i;m_0,m_j)$$

using the formula

$$r(l_1,l_2,l_3,l_4) = \frac{\Delta(l_1,l_3)\Delta(l_2,l_4)}{\Delta(l_1,l_4)\Delta(l_2,l_3)}$$

for the cross-ratio. We will get

$$\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_0,...,\hat{m}_j,...,m_n) \tag{2}$$

$$-\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_0,...,\hat{m}_j,...,m_n) \tag{3}$$

$$-\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_1,...,m_n) \tag{4}$$

$$+\sum_{i,j=1}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_1,...,m_n) \tag{5}$$

Applying to (2) dual additivity in L we can rewrite it as

$$-\sum_{j=1}^{n}(-1)^{j}(l_0|l_1,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_0,m_0,...\hat{m}_j,...,m_n)$$

(3) is already in the desired shape. Applying to (4) first additivity in M and then dual additivity in L we get

$$-(l_0|l_1,...,l_n;m_1,...,m_n)\otimes\Delta(l_0,m_1,...,m_n)$$

Finally applying additivity in M to (5) we get

$$-\sum_{i=1}^{n}(-1)^i(l_i|l_0,...,\hat{l}_i,...,l_n;m_1,...,m_n)\otimes\Delta(l_i,m_1,...,m_n)$$

So we conclude that (1) is equal to

$$-\sum_{i,j=0}^{n}(-1)^{i+j}(l_i|l_0,...,\hat{l}_i,...,l_n;m_0,...,\hat{m}_j,...,m_n)\otimes\Delta(l_i,m_0,...,\hat{m}_j,...,m_n)$$

The considerations for $\nu_{1,n-1}$ are similar. The proposition is proved.

3 Main construction

1. The weight two case. Let $B_2(F)$ be the quotient of the free abelian group $\mathbb{Z}[F^*\backslash\{1\}]$ generated by the symbols $\{x\}$ where $x\in F^*\backslash\{1\}$ modulo the subgroup $R_2(F)$ generated by the "five term relations", i.e. by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(x_1,...,\hat{x}_i,...,x_5)\},\qquad x_i\in\mathbb{P}^1(F),\quad x_i\neq x_j$$

Denote by $\{x\}_2$ the image of the generator $\{x\}$ in $B_2(F)$. One can prove (see for example [G1]) that there is a well defined homomorphism

$$\delta_2: B_2(F)\longrightarrow\Lambda^2F^*,\quad \{x\}_2\longmapsto(1-x)\wedge x$$

The complex we get is called the Bloch complex.

In [BGSV1-2] there was defined a homomorphism of complexes

$$\begin{array}{ccc} A_2 & \stackrel{\nu}{\longrightarrow} & A_1\otimes A_1\\ \downarrow a_2 & & \downarrow a_1\wedge a_1\\ B_2 & \stackrel{\delta_2}{\longrightarrow} & \Lambda^2F^*\end{array}$$

Namely

$$a_2(l_0,l_1,l_2;m_0,m_1,m_2):=< l_0,l_1,l_2;m_0,m_1,m_2>_2:= \qquad (6)$$
$$\sum_{i,j=0}^{2}(-1)^{i+j}\{r(l_i|l_0,...,\hat{l}_i,...,l_2;m_0,...,\hat{m}_j,...,m_2)\}_2\in B_2(F)$$

2. The weight three motivic complex related to the classical trilogarithm. Let V_3 be a three dimensional vector space over a field F. Choose a volume form $\omega_3 \in det V_3^*$. Recall that for any three vectors l_1, l_2, l_3 in V_3 we have defined the "determinant"

$$\Delta(l_1, l_2, l_3) := < l_1 \wedge l_2 \wedge l_3, \omega_3 >$$

Let us define the generalized cross-ratio of six generic points $x_1, ..., x_6$ on the plane $\mathbb{P}^2$ by setting

$$r_3(x_1, ..., x_6) := \frac{1}{15}\text{Alt}_6\{\frac{\Delta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_4)\Delta(\tilde{x}_2, \tilde{x}_3, \tilde{x}_5)\Delta(\tilde{x}_3, \tilde{x}_1, \tilde{x}_6)}{\Delta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_5)\Delta(\tilde{x}_2, \tilde{x}_3, \tilde{x}_6)\Delta(\tilde{x}_3, \tilde{x}_1, \tilde{x}_4)}\} \in \mathbb{Z}[F^*\backslash\{1\}] \tag{7}$$

Here $\tilde{x}$ is a vector projecting to the point x. The ratio does not depend on the choice of these vectors.

The group $R_3(F)$ of functional equations for the trilogarithm. I will use the following definition, which is a bit ad hoc. $R_3(F)$ is the group generated by $\{x\}_3 - \{x^{-1}\}_3$, the "seven term" relations (containing actually 7! terms)

$$\sum_{i=1}^{7}(-1)^i r_3(x_1, ..., \hat{x}_i, ..., x_7), \qquad x_i \in \mathbb{P}^2(F) \tag{8}$$

where the points $x_1, ..., x_7$ are in *generic* position in the plane, and Kummer's functional equation for the trilogarithm:

$$K(x,y) := -\{\frac{x(1-y)^2}{y(1-x)^2}\} - \{xy\} - \{\frac{x}{y}\} - 2\{1\} +$$

$$2\Big(\{\frac{-x(1-y)}{(1-x)}\} + \{\frac{x(1-y)}{y(1-x)}\} + \{\frac{-y(1-x)}{1-y}\} + \{\frac{1-x}{1-y}\} + \{y\} + \{x\}\Big)$$

It might be true that Kummer's relation follows from the generic seven term relations and the one $\{x\}_3 - \{x^{-1}\}_3$. But I just add them to the list.

Remark. A more natural way to define the group $R_3(F)$ is this. We extend the generalized cross-ratio to arbitrary configurations of 6 points on the plane. Then $R_3(F)$ is given by the seven term relations for arbitrary configurations of seven points in the plane. Using the main results of [G0-G1] one can show that both definitions lead to the same group. In particular we get Kummer's relations for a certain degenerate configuration of seven points. To extend the definition of r_3 we take the "limit value" of the definition (7) using $\{0\}_3 = \{\infty\}_3 = 0$. If two of the points x_i coincide or four of them are on a line then $r_3(x_1, ..., x_6) = 0$. For the remaining cases see lemma 3.7 below. To make the exposition shorter I will not use this approach.

Set

$$B_3(F) := \frac{\mathbb{Z}[F^*\backslash\{1\}]}{R_3(F)}$$

Denote by $\{x\}_3$ the image in $B_3(F)$ of the generator $\{x\}$. One can prove ([G1-3]) that there is a well defined homomorphism

$$\delta_3 : B_3(F) \longrightarrow B_2(F) \otimes F^*, \quad \{x\}_3 \longmapsto \{x\}_2 \otimes x$$

We get the weight three motivic complex related to the classical trilogarithm

$$B_3(F) \longrightarrow B_2(F) \otimes F^* \longrightarrow \Lambda^3 F^*$$

where $\{x\}_2 \otimes y \longmapsto (1-x) \wedge x \wedge y$.

Lemma 3.1 *Using the notations introduced in (6)*

$$\begin{gathered}\delta_3 \circ r_3(l_1, l_2, l_3, l_4, l_5, l_6) = -\frac{1}{18}\mathrm{Alt}_6\Big(< l_1, l_2, l_3; l_4, l_5, l_6 >_2 \otimes \Delta(l_1, l_2, l_3)\Big) = \\ -2 < l_1, l_2, l_3; l_4, l_5, l_6 >_2 \otimes \Delta(l_1, l_2, l_3) + \quad 19 \textit{ other terms}\end{gathered} \tag{9}$$

Proof. Let $C_n(V_3)$ be the free abelian group generated by the configurations of n generic vectors (i.e. n-tuples of vectors in generic position modulo the action of $GL(V_3)$) in V_3. Let

$$d : C_n(V_3) \to C_{n-1}(V_3), \quad d(l_1, ..., l_n) := \sum_{i=1}^{n} (-1)^{i-1}(l_1, ..., \hat{l}_i, ..., l_n)$$

Define a homomorphism $C_5(V_3) \to B_2(F) \otimes F^*$ by setting

$$f_5(3)(l_1, ..., l_5) := \frac{1}{2}\mathrm{Alt}_5\Big(\{r(l_1|l_2, ..., l_5)\}_2 \otimes \Delta(l_1, l_2, l_3)\Big)$$

According to theorem 2.3 in the Appendix to [G3] the following diagram is commutative:

$$\begin{array}{ccc} C_6(V_3) & \stackrel{d}{\longrightarrow} & C_5(V_3) \\ r_3 \downarrow & & \downarrow f_5(3) \\ B_3(F) & \stackrel{\delta_3}{\longrightarrow} & B_2(F) \otimes F^* \end{array}$$

Therefore

$$\delta_3 \circ r_3(l_1, l_2, l_3, l_4, l_5, l_6) = -\frac{1}{2}\mathrm{Alt}_6\Big(\{r(l_1|l_2, l_3, l_4, l_5)\}_2 \otimes \Delta(l_1, l_2, l_3)\Big)$$

which is equal to (9).

3. A homomorphism between the weight three motivic complexes. We have constructed in chapter 2 a complex

$$A_3^0 \stackrel{\nu_{2,1} \oplus \nu_{1,2}}{\longrightarrow} A_2 \otimes A_1 \oplus A_1 \otimes A_2 \stackrel{\nu_{1,1} \otimes Id - Id \otimes \nu_{1,1}}{\longrightarrow} A_1 \otimes A_1 \otimes A_1$$

Let us define a homomorphism of complexes

$$
\begin{array}{ccccc}
A_3^0 & \xrightarrow{\nu} & A_2 \otimes A_1 \oplus A_1 \otimes A_2 & \longrightarrow & A_1 \otimes A_1 \otimes A_1 \\
\downarrow a_3 & & \downarrow a_2 \wedge a_1 & & \downarrow \wedge^3 a_1 \\
B_3 & \xrightarrow{\delta_3} & B_2(F) \otimes F^* & \longrightarrow & \Lambda^3 F^*
\end{array}
$$

where $a_2 \wedge a_1(x_2 \otimes x_1 + y_1 \otimes y_2) := a_2(x_2) \otimes a_1(x_1) - a_2(y_2) \otimes a_1(y_1)$ and $\wedge^3 a_1(x_1 \otimes x_2 \otimes x_3) = a_1(x_1) \wedge a_1(x_2) \wedge a_1(x_3)$.

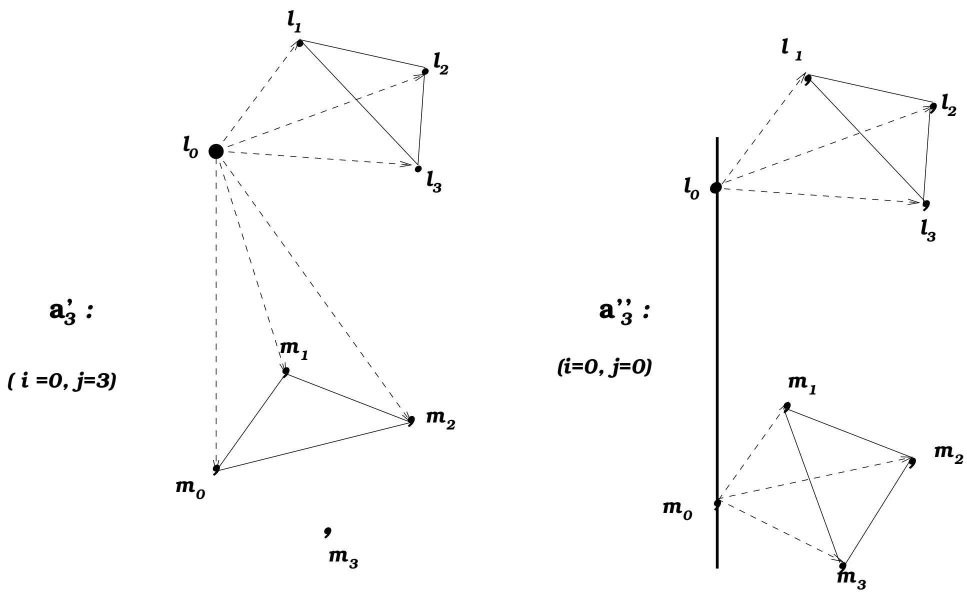


figure 3

Let $x_1, x_2, x_3; y_1, y_2, y_3$ be six different points on a line. Set

$$\mu_3(x_1, x_2, x_3; y_1, y_2, y_3) =$$
$$\frac{1}{4} \cdot \mathrm{Alt}_{(x_1,x_2,x_3)(y_1,y_2,y_3)} \Big(\{r(x_1, y_2, x_2, y_1)\}_3 - \{r(x_1, y_1, x_2, y_2)\}_3 \Big)$$

This formula, which contains 18 terms, means simply that $\mu_3(x_1, x_2, x_3; y_1, y_2, y_3)$ is skewsymmetric with respect to x_1, x_2, x_3 and y_1, y_2, y_3 and a typical term is $\{r(x_1, y_2, x_2, y_1)\}_3$.

We set

$$a_3 := \frac{1}{6} a'_3 - \frac{1}{3} a''_3$$

where a_3' and a_3'' are defined on generators by the following formula (see fig. 3):

$$a_3'(l_0,l_1,l_2,l_3;m_0,m_1,m_2,m_3) := \sum_{i,j=0}^{3}(-1)^{i+j}r_3(l_i|l_0,...,\hat{l}_i,...,l_3;m_0,...,\hat{m}_j,...,m_3)$$

$$a_3''(l_0,...,l_3;m_0,...,m_3) := \sum_{i,j=0}^{3}(-1)^{i+j}\cdot\mu_3(l_i,m_j|l_0,...,\hat{l}_i,...,l_3;m_0,...,\hat{m}_j,...,m_3)$$

Denote by $\tilde{A}_3^0$ the free abelian group generated by the generators of the group A_3^0, i.e. by pairs of tetrahedra in generic position.

Theorem 3.2 *The diagram*

$$\begin{array}{ccc} \tilde{A}_3^0 & \stackrel{\nu}{\longrightarrow} & A_2^0\otimes A_1^0\oplus A_1^0\otimes A_2^0 \\ \downarrow a_3 & & \downarrow a_2\wedge a_1 \\ B_3 & \stackrel{\delta_3}{\longrightarrow} & B_2(F)\otimes F^* \end{array}$$

is commutative.

Remark. The proof of this theorem is the crucial point of the paper. The commutativity of the diagram is what we really wanted for the homomorphism a_3. Surprisingly neither homomorphism a_3' nor a_3'' make the diagram commutative, even up to a scalar. Only their sum does the job. Moreover, theorem (3.2) "morally" implies that a_3 should be a homomorphism of groups. We will prove this later on.

Proof. Because of the skew-symmetry of the formulas for ν (see proposition 2.3) it is sufficient to compute for

$$(a_2\wedge a_1)\circ\nu(l_0,...,l_3;m_0,...,m_3) \quad \text{and} \quad \delta_3\circ a_3(l_0,...,l_3;m_0,...,m_3)$$

the $B_2(F)$-factors of the following elements of F^*:

$$\Delta(m_0,m_1,m_2,m_3), \quad \Delta(l_0,m_0,m_1,m_2), \quad \Delta(l_0,l_1,m_0,m_1),$$

$$\Delta(l_0,l_1,l_2,m_0), \quad \Delta(l_0,l_1,l_2,l_3)$$

Since $(a_2\wedge a_1)\circ\nu(L,M) = (a_2\wedge a_1)\circ\nu(M,L)$ by the definition of a_2 and a_1 and by proposition 2.3, and $a_3(L;M) = a_3(M;L)$ by lemma (3.5), we see that it is sufficient to consider only the first three of them.

It follows from proposition (2.3) that in $(a_2\wedge a_1)\circ\nu(l_0,...,l_3;m_0,...,m_3)$ appears only

$$< l_0|l_1,l_2,l_3;m_1,m_2,m_3 >_2 \otimes\Delta(l_0,m_1,m_2,m_3) \tag{10}$$

Step 1. Let us do the computations for $\delta_3 \circ a_3(l_0,...,l_3;m_0,...,m_3)$. The crucial and most nontrivial case is the term with $\Delta(l_0,l_1,m_0,m_1)$. If the diagram is commutative it must be zero because of the observation we just made. The only summands in $a_3'(l_0,...,l_3;m_0,...,m_3)$ that give a contribution to this term are:

$$-r_3(l_0|l_1,l_2,l_3,m_0,m_1,m_2)+r_3(l_0|l_1,l_2,l_3,m_0,m_1,m_3)+$$
$$r_3(l_1|l_0,l_2,l_3,m_0,m_1,m_2)-r_3(l_1|l_0,l_2,l_3,m_0,m_1,m_3)$$

Lemma (3.1) shows using the symmetry $1<->4, 2<->5$ that the term with $\Delta(l_0,l_1,m_0,m_1)$ in $\delta_3 \circ a_3(l_0,...,l_3;m_0,...,m_3)$ is

$$-2\Big(-<l_0|l_1,m_0,m_1;l_2,l_3,m_2>_2+<l_0|l_1,m_0,m_1;l_2,l_3,m_3>_2+$$

$$<l_1|l_0,m_0,m_1;l_2,l_3,m_2>_2-<l_1|l_0,m_0,m_1;l_2,l_3,m_3>_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1)=$$

$$2\cdot \mathrm{Alt}_{l_0l_1}\mathrm{Alt}_{m_2m_3}\Big(<l_0|l_1,m_0,m_1;l_2,l_3,m_2>_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1) \qquad (11)$$

Step 2. The contribution for $\Delta(l_0,l_1,m_0,m_1)$ in $\delta_3 \circ a_3''(l_0,...,l_3;m_0,...,m_3)$ is coming from

$$\delta_3\circ\mu_3\Big((l_0m_0|l_1,l_2,l_3;m_1,m_2,m_3)-(l_0m_1|l_1,l_2,l_3;m_0,m_2,m_3)$$
$$-(l_1m_0|l_0,l_2,l_3;m_1,m_2,m_3)+(l_1m_1|l_0,l_2,l_3;m_0,m_2,m_3)\Big)$$

And the contribution is equal to

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3);(m_0,m_1);(l_2,l_3)}\Big((l_0m_0|l_1,m_1,l_2,m_2)_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1)=$$

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3)}\Big(<l_0|m_0,l_1,m_1;l_2,m_2,l_3>_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1)=$$

$$\mathrm{Alt}_{(l_0,l_1);(m_2,m_3)}\Big(<l_0|l_1,m_0,m_1;l_2,l_3,m_2>_2\Big)\otimes\Delta(l_0,l_1,m_0,m_1)$$

(We use a shorthand $(l_0m_0|l_1,m_1,l_2,m_2)_2$ for $\{r(l_0m_0|l_1,m_1,l_2,m_2)\}_2$). Comparing the last formula with (11) we see that

$$\tfrac{1}{2}a_3'-a_3'' \textit{ has contribution } 0\otimes\Delta(l_0,l_1,m_0,m_1)$$

Step 3. Now look at the terms of $\delta_3\circ a_3(l_0,...,l_3;m_0,...,m_3)$ with $\Delta(m_0,m_1,m_2,m_3)$ and $\Delta(l_0,m_1,m_2,m_3)$.

1) $\Delta(m_0,m_1,m_2,m_3)$ does not appear in $a_3(l_0,...,l_3;m_0,...,m_3)$.

2i)Using proposition (2.3) we get that $\Delta(l_0,m_1,m_2,m_3)$ appears in $\delta_3\circ a_3'(l_0,...,l_3;m_0,...,m_3)$ as

$$2<l_0|l_1,l_2,l_3;m_1,m_2,m_3>_2\otimes\Delta(l_0,m_1,m_2,m_3)$$

2ii) Let us compute $\delta_3 \circ a_3''(l_0, ..., l_3; m_0, ..., m_3)$. There are only 3 terms in a_3'' where the term $\Delta(l_0, m_1, m_2, m_3)$ appears; we can write them as

$$-1/2\mathrm{Alt}_{(m_1,m_2,m_3)}\mu_3(l_0m_1|l_1, l_2, l_3; m_0, m_2, m_3)$$

It is equal to

$$1/2\mathrm{Alt}_{(l_1,l_2,l_3)(m_1,m_2,m_3)}(l_0m_1|l_1, m_2, l_2, m_3)_2 \otimes \Delta(l_0, m_1, m_2, m_3) =$$
$$-2 < (l_0|m_1, m_2, m_3; l_1, l_2, l_3) >_2 \otimes\Delta(l_0, m_1, m_2, m_3)$$

For the last step use $(l_0m_1|l_1, m_2, l_2, m_3)_2 = -(l_0m_1|m_2, m_3, l_1, l_2)_2$. So the contribution of $\frac{1}{2}a_3' - a_3''$ is

$$3\cdot < l_0|l_1, l_2, l_3; m_1, m_2, m_3 >_2 \otimes\Delta(l_0, m_1, m_2, m_3)$$

It remains to compare this answer with (10). Theorem (3.2) is proved.

Theorem 3.3 *a_3 is a homomorphism of groups.*

We start the proof with

Proposition 3.4 *Both a_3' and a_3'' send relations 2), 3), 4) to zero.*

Proof. This is clear for relations 2) and 4). It is also clear that a_3'' is additive in L and in M. To check that a_3'' sends the dual additivity in L to zero notice that the typical term for $a_3''(\sum_{i=0}^{3}(-1)^i \cdot (l_i|l_0, ..., \hat{l}_i, ..., l_4; m_0, ..., m_3))$ is

$$\pm\mu_3(l_il_jm_k|l_0, ..., \hat{l}_i, ..., \hat{l}_j, ..., l_4; m_0, ..., \hat{m}_k, ..., m_3)$$

It is symmetric in l_i, l_j. So we get 0 after alternation in $(l_0, ..., l_4)$.

Similarly using the skewsymmetry of r_3 we immediately see that the map a_3' sends dual additivity in L to zero. The additivity in M of a_3' is obvious. Therefore the proposition follows immediately from the following lemma.

Lemma 3.5 $a_3'(L; M) = a_3'(M; L); \qquad a_3''(L; M) = a_3''(M; L)$

Proof. The statement about a_3'' is clear from the definition. Let us prove it for a_3'. Applying the 7-term relation for the configuration of 7 points in $\mathbb{P}^2$ $(l_i|l_0, ..., \hat{l}_i, ..., l_3, m_0, ..., m_3)$ we get

$$a_3'(l_0, , ..., l_3; m_0, ..., m_3) = -\sum_{0\leq i\neq j\leq 3}\gamma(i, j)r_3(l_i|l_0, ..., \hat{l}_i, ..., \hat{l}_j, ..., l_3, m_0, ..., m_3) \tag{12}$$

where $\gamma(i, j) = (-1)^{i+j}$ if $i < j$ and $-(-1)^{i+j}$ otherwise. Applying the dual 7-term relation to the configuration $(l_0, ..., \hat{l}_i, ..., l_3, m_0, ..., m_3)$ of 7 points in $\mathbb{P}^3$ we can rewrite (12) as

$$-\sum_{i,j=0}^{3}(-1)^{i+j}r_3(m_i|l_0, ..., \hat{l}_j, ..., l_3, m_0, ..., \hat{m}_i, ..., m_3)$$

It remains to interchange the l- and m- tetrahedra using the skewsymmetry. The lemma is proved.

Neither a_3' nor a_3'' send the relation 1) to zero. Only their weighted sum a_3 does the job. To show this we have to prove the proposition 3.6 below.

Let $L_0, ..., L_3$ be 4 lines and $m_0, ..., m_3$ 4 points in $\mathbb{P}^2$. Let

$$(L_0, ..., \hat{L}_i, ..., L_3; m_0, ..., \hat{m}_j, ..., m_3)$$

be a pair of triangles where the first triangle is given by its sides and the second one by its vertices (see fig. 4). For example

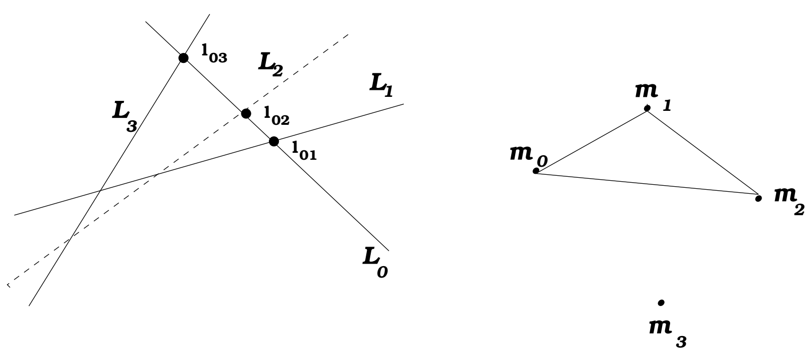
$$(L_1, L_2, L_3; m_1, m_2, m_3) = (l_{23}, l_{13}, l_{12}, m_1, m_2, m_3)$$

where $l_{ij} := L_i \cap L_j$, so the first three points are the vertices of the triangle (L_1, L_2, L_3).

Proposition 3.6 *Let $L_0, ..., L_3$ be 4 lines and $m_0, ..., m_3$ 4 points in $\mathbb{P}^2$. Then*

$$\sum_{i,j=0}^{3} (-1)^{i+j} r_3(L_0, ..., \hat{L}_i, ..., L_3; m_0, ..., \hat{m}_j, ..., m_3) = \tag{13}$$

$$2 \sum_{i,j=0}^{3} (-1)^{i+j} \mu_3(m_i | l_{j0}, ..., \hat{l}_{jj}, ..., l_{j3}; m_0, ..., \hat{m}_i, ..., m_3) \tag{14}$$



the pair of triangles $(L_0, L_1, L_3; m_0, m_1 m_2)$

figure 4

Proof. Applying the 7-term relation to the configuration of 7 points formed by the vertices of the triangle $L_0, ..., \hat{L}_j, ..., L_3$ and $m_0, ..., m_3$ we rewrite (13) as

$$-\frac{1}{2}\mathrm{Alt}_{(L_0,...,L_3)} r_3(l_{13}, l_{23}, m_0, m_1, m_2, m_3)$$

(We alternate the indices $0,1,2,3$ in the l-variables only). It is equal to

$$-r_3(l_{13},l_{23},m_0,m_1,m_2,m_3)+r_3(l_{03},l_{23},m_0,m_1,m_2,m_3)-r_3(l_{03},l_{13},m_0,m_1,m_2,m_3)$$

$$+ 9 \text{ other terms}$$

Consider the three intersection points of the line L_i with the other lines and add to them the points m_0,m_1,m_2,m_3. For instance $(l_{03},l_{13},l_{23},m_0,m_1,m_2,m_3)$ is the configuration related to the line L_3. Applying the 7-term relation to these configurations we rewrite the previous formula as

$$-r_3\Big(\sum_{i=0}^{3}(-1)^i(l_{03},l_{13},l_{23},m_0,...,\hat{m}_i,...,m_3)-\sum_{i=0}^{3}(-1)^i(l_{10},l_{20},l_{30},m_0,...,\hat{m}_i,...,m_3) \tag{15}$$

$$+\sum_{i=0}^{3}(-1)^i(l_{21},l_{31},l_{01},m_0,...,\hat{m}_i,...,m_3)-\sum_{i=0}^{3}(-1)^i(l_{32},l_{02},l_{12},m_0,...,\hat{m}_i,...,m_3)\Big) \tag{16}$$

Now comes a little trick: we will use the fact that one can extend the generalized cross ratio to certain degenerate configurations of six points such that the seven term relation holds.

Lemma 3.7 *Suppose the points l_1,l_2,l_3 are on the same line. Then*

$$r_3(l_1,l_2,l_3,m_1,m_2,m_3) = -2\mu_3(l_1,l_2,l_3;m_1,m_2,m_3)$$

Remark. It is easy to check that

$$\delta_3 \circ (r_3+2\mu_3)(l_1,l_2,l_3,m_1,m_2,m_3) = 0$$

This implies that $(r_3+2\mu_3)(l_1,l_2,l_3,m_1,m_2,m_3)$ is a functional equation for the trilogarithm. The following proof verifies that it is already in the group R_3 defined above (so we do not have to enlarge this group).

Proof. The following proof based on direct computation using $\{0\}_3 = \{\infty\}_3 = 0$ was provided to me by Herbert Gangl. Consider the following special configuration of 6 points on the plane (fig. 5)

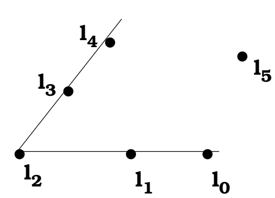


figure 5

given by the columns of the matrix

$$\mathcal{C}(b,c) := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & c & 1 & b & 1 & 0 \end{pmatrix}$$

It is sufficient to prove the lemma for this configuration, since the general case will follow by the seven term relation.

Set

$$\tilde{l}_3\{x\} := \{1-x^{-1}\}_3 - \{1-x\}_3$$

$$M_3(\mathcal{C}(b,c)) := \tilde{l}_3\Big(\{\frac{b-c}{1-c}\}_3 - \{b\}_3 + \{\frac{b(1-c)}{b-c}\}_3 - \{1-c\}_3 + \{\frac{b-c}{b}\}_3\Big)$$

A configuration of six points $x_1, x_2, x_3, l_1, l_2, l_3$ on the plane, three of which, x_1, x_2, x_3, are on a line L, is determined completely by the configuration $(x_1, x_2, x_3, n_1, n_2, n_3)$ of $3+3$ points on the line L, where the point $n_1 := l_2 l_3 \cap L$ and so on, see fig. 6.

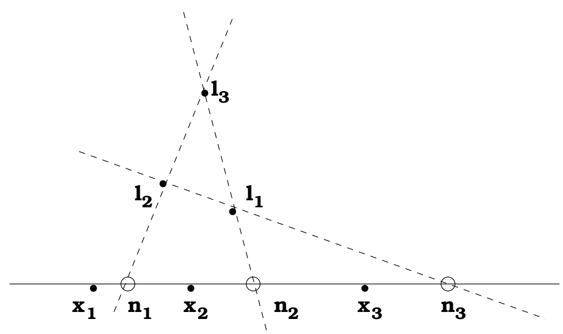


figure 6

(Recall that by configuration we always mean its projective equivalence class.)

Let $\tilde{\mathcal{C}}(b,c)$ be the configuration of six points on the line corresponding to the configuration $\mathcal{C}(b,c)$ by this rule. Then

$$M_3(\mathcal{C}(b,c)) = -\mu_3(\tilde{\mathcal{C}}(b,c))$$

One has

$$15 \cdot r_3(\mathcal{C}(b,c)) = -18\{1-b\}_3 - 12\{\frac{b}{1-c}\}_3 - 18\{\frac{b-c}{b-1}\}_3 - 18\{c\}_3$$

$$-12\{\frac{1-b}{c}\}_3 - 24\{\frac{b(c-1)}{c(b-1)}\}_3 - 6\{\frac{c}{(c-b)(1-b)}\}_3 - 6\{\frac{c(b-c)}{b-1}\}_3 - 18\{\frac{c-b}{c}\}_3$$

$$+18\{1-b^{-1}\}_3 + 12\{b-c\}_3 + 18\{\frac{b-1}{c-1}\}_3 + 12\{\frac{b-1}{c}\}_3 + 18\{\frac{b}{c}\}_3$$

$$+24\{\frac{b-c}{c(b-1)}\}_3+6\{\frac{b(b-1)}{c(c-1)}\}_3+18\{1-c^{-1}\}_3+6\{\frac{(1-b)(1-c)}{b\cdot c}\}_3$$

Recall two Kummer relations:

$$6K(\frac{b}{1-c},\frac{c}{1-b})=12\{\frac{b}{1-c}\}_3+12\{\frac{c}{1-b}\}_3+12\{\frac{1-b}{1-c}\}_3+12\{\frac{c}{c-1}\}_3+12\{\frac{b}{b-1}\}_3$$

$$+12\{\frac{c}{b}\}_3-6\{\frac{bc}{(1-b)(1-c)}\}_3-6\{\frac{b(b-1)}{c(c-1)}\}_3-6\{\frac{c(1-b)}{b(1-c)}\}_3-12\{1\}_3$$

and

$$-6K(b-c,\frac{b-1}{c})=-12\{b-c\}_3-12\{\frac{b-1}{c}\}_3-12\{c\}_3-12\{1-b\}_3-12\{\frac{c}{c-b}\}_3$$

$$-12\{\frac{b-1}{b-c}\}_3+6\{\frac{(c-b)(1-b)}{c}\}_3+6\{\frac{c(b-c)}{b-1}\}_3+6\{\frac{c(b-1)}{b-c}\}_3+12\{1\}_3$$

Then one computes, adding the three expressions above,

$$15r_3(\mathcal{C}(b,c))+6K(\frac{b}{1-c},\frac{c}{1-b})-6K(b-c,\frac{b-1}{c})=$$

$$30\Big(-\{1-b\}_3+\{\frac{b}{b-1}\}_3-\{\frac{b-c}{b-1}\}_3+\{\frac{1-b}{1-c}\}_3-\{\frac{c(1-b)}{b(1-c)}\}_3+\{\frac{c-b}{c(1-b)}\}_3$$

$$-\{c\}_3+\{1-c^{-1}\}_3-\{\frac{c-b}{c}\}_3+\{\frac{c}{b}\}_3\Big)=-30\mu_3(\mathcal{C}(b,c))$$

which is exactly what we wanted. The lemma is proved. Comparing this lemma with (15), (16) we get proposition 3.4 and hence theorem (3.3).

Another way to proceed. The right hand side of (13) considered as a function of 4 lines and 4 points on the plane satisfies the 5 term relation with respect to lines as well as points. Using this observation it is sufficient to check the proposition for the degenerate configuration of lines and points when 3 of the lines L_i pass through a point and 3 points among the m_i are on a line (see fig. 7).

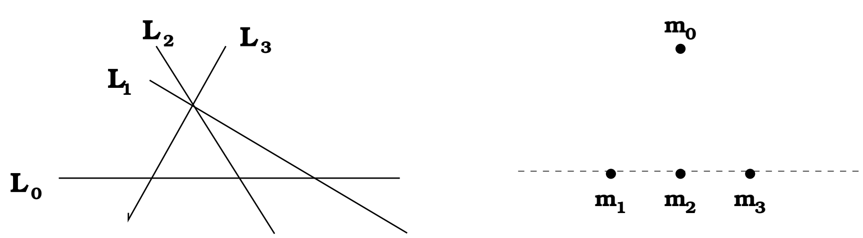


figure 7

4. Another way to define the homomorphism a_3. Let us associate to 8 points $(l_0, ..., l_3; m_0, ..., m_3)$ in $\mathbb{P}^3$ a degenerate configuration of 6 points on the plane, denoted

$$d(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)$$

as follows. Consider the configuration

$$(l_i|l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., m_3)$$

of 7 points in $\mathbb{P}^2$. We construct out of them a degenerate configuration of 6 lines in the plane. Let L_{ij} (resp (M_{ij}) be the line through the points l_i and l_j (resp. m_i and m_j). Take the three lines formed by the sides of the triangle $(l_i|l_0, ..., \hat{l}_i, ..., l_3)$ and add to them the three lines $M_{j0}, ...\hat{M}_{jj}, ..., M_{j3}$ connecting m_j with the other three m-points. For example

$$d(l_0, m_0||l_1, l_2, l_3; m_1, m_2, m_3) = (L_{23}, L_{13}, L_{12}, M_{01}, M_{02}, M_{03})$$

Let $\bar{C}_{2n}(F)$ be the free abelian group generated by the configurations of *arbitrary* $2n$ F-points in $\mathbb{P}^{n-1}$. One can define a_3 as a composition

$$A_3^0(F) \xrightarrow{p_3} \bar{C}_6(F)/\{7 \text{ term relations}\} \xrightarrow{r_3} B_3(F)$$

where (see fig. 8)

$$p_3(l_0, l_1, l_2, l_3; m_0, m_1, m_2, m_3) := \sum_{i,j=0}^{3} \Big((-1)^{i+j}(l_i|l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)$$

$$+ 2 \cdot d(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)\Big) \qquad (17)$$

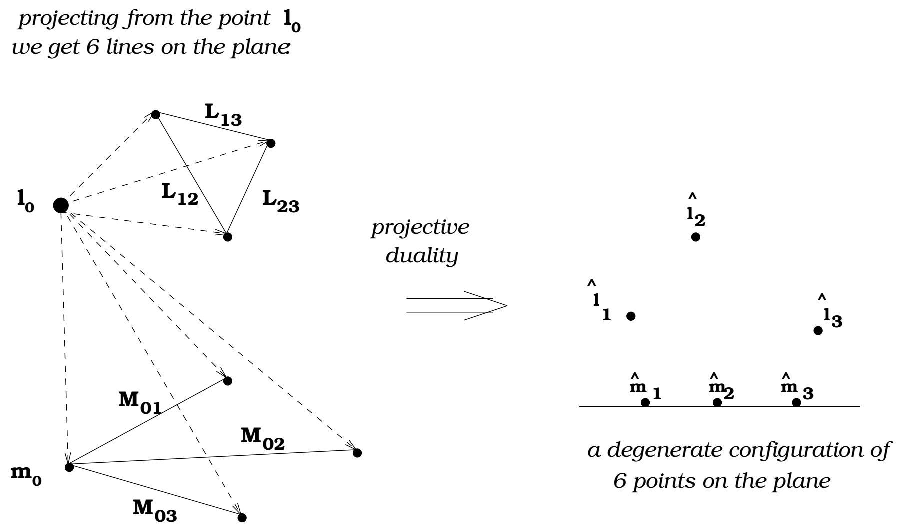


figure 8

There is a different candidate for the term (17). We define a configuration

$$\hat{d}(l_i, m_j || l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3) \tag{18}$$

of $3+3$ points on a line as follows. Consider the planes L_i and M_j in $\mathbb{P}^3$ (see fig. 9 for $i = 0, j = 0$.) Their intersection is a line $L_i \cap M_j$. The sides of the triangles $(l_0, ..., \hat{l}_i, ..., l_3)$ and $(m_0, ..., \hat{m}_j, ..., m_3)$ cut this line in $3+3$ points. This is the configuration (18) we promised to define.

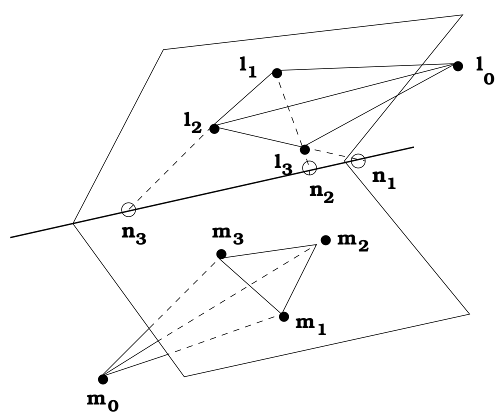


figure 9

Recall that we can think of a configuration of $3+3$ points on a line as of a configuration of six points on a plane (see fig. 6). So we can describe configu-

ration (18) by the configuration of points $(n_0, ..., \hat{n}_i, ..., n_3, m_0, ..., \hat{m}_j, ..., m_3)$ on the plane M_j, or by a similar configuration on the plane L_i.

The definition of (18) given on fig. 9 is projectively dual to the one on fig. 8. This means that if we consider the points l_i, m_j as planes in the dual space $\hat{\mathbb{P}}^3$ then (17) corresponds to (18).

Remark. One has (mainly thanks to lemma (3.7))

$$\sum_{i,j=0}^{3}(-1)^{i+j}\Big(d(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)- \tag{19}$$

$$\hat{d}(l_i, m_j||l_0, ..., \hat{l}_i, ..., l_3; m_0, ..., \hat{m}_j, ..., m_3)\Big) = 0$$

Let us imagine that we would be able to prove this using only the (possibly degenerate) seven term relations but not lemma (3.7). Then it is straightforward to show that p_3 sends to zero all the defining relations for the group A_3 *except* the degeneracy relation. So we can define the group R_3 by adding to the (possibly degenerate) seven term relations the image of the degeneracy relations, i.e. the relations from proposition (3.6). Moreover we get a nice free gift: now we can skip lemma (3.7) together with its computational proof. So instead of the mysterious relation (8) for the classical trilogarithm which we think of as a relation

$$\text{generic configuration of 6 points on } \mathbb{P}^2 \quad = \sum \quad \text{of degenerate configurations}$$

(and which *does not* follow from the seven term relations between the configurations of six points on the plane), we would have the geometrically natural relations from proposition (3.6). However I was not able to prove (19) without using lemma (3.7). I wish somebody will try.

5. The key ansatz from [G0-3]. Consider the following admissible pair of tetrahedra which represents 0 on A_3 (see fig 10).

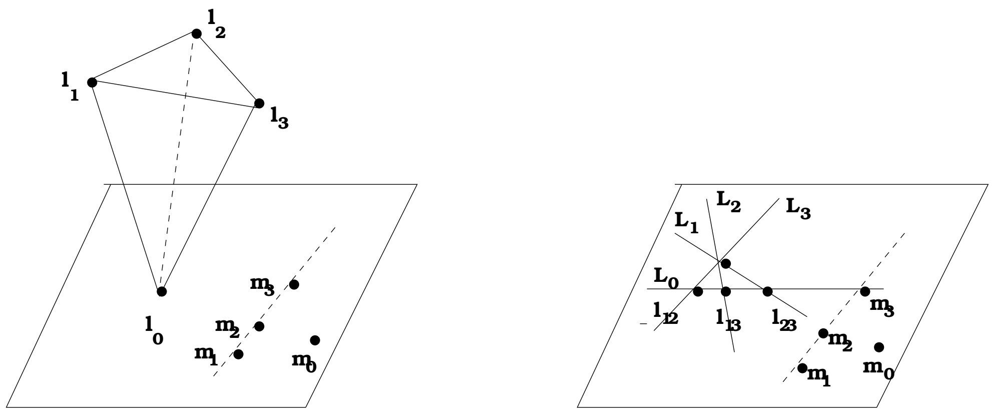


figure 10

Here l_0, m_0, m_1, m_2, m_3 are on the same plane. The right hand side of the picture illustrates the computation of the homomorphism a_3.

Then

$$a_3'(L, M) = \{r_3(l_{12}, l_{13}, l_{23}, m_1, m_2, m_3)\}$$

Indeed, all the other terms in (15)-(16) vanish since by definition a degenerate configuration of six points on the plane represents zero in B_3 if two of them coincide or four are on the same line. The condition that for such (L, M) one has $6a_3(L, M) = (a_3' - 2a_2'')(L, M) = 0$ just means that we get a formula expressing the configuration $l_{12}, l_{13}, l_{23}, m_1, m_2, m_3$ as a sum of configurations like the one on fig. 11 (each of them corresponds to a generator $\{x\}_3$ of the group B_3).

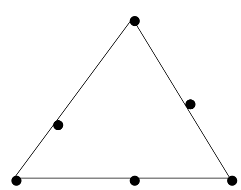


figure 11

6. A formula for $\Lambda_3(L, M)$. One can define (see [BGSV]) a commutative product map

$$\mu : A_k \otimes A_l \to A_{k+l}$$

Then one should have a structure of a Hopf algebra on $A_\bullet$ given by the coproduct ν and the product μ (however at the present moment it is not clear how to define

the coproduct for the degenerate admissible pairs of simplices, and thus we can not prove that $\nu \circ \mu = \mu \otimes \mu(\nu))$. Let

$$P_n := \oplus_{k+l=n} \mu(A_k \otimes A_l) \qquad (k > 0, l > 0)$$

Consider the homomorphism $\pi_3 : A_3 \to A_3$ given by

$$\pi_3(x) := x - \frac{1}{2}\mu \circ \nu(x) + \frac{1}{3}\mu \circ \nu_{1,1,1}(x)$$

where $\nu_{1,1,1} : A_3 \to A_1 \otimes A_1 \otimes A_1$ is the composition

$$A_3 \xrightarrow{\nu_{2,1}} A_2 \otimes A_1 \xrightarrow{\nu_{1,1} \otimes id} A_1 \otimes A_1 \otimes A_1$$

Lemma 3.8 *π_3 is a projector and $Ker\pi_3 = P_3$.*

See also section 4.3 for a generalization.

Proof. Let us check that $P_3 \subset Ker\pi_3$. This implies $\pi_3^2 = \pi_3$. Since $\pi_3 = id$ on A_3/P_3 we have $Ker\pi_3 = P_3$.

Suppose $\nu_{1,1}(y_2) = \sum_i y_i' \otimes y_i'' \subset A_1 \otimes A_1$. Let $x_1 \in A_1$. Denote by $\cdot$ the product in $A_\bullet$. Then

$$\nu \circ \mu(y_2 \cdot x_1) = \sum_i (x_1 \cdot y_i' \otimes y_i'' + y_i' \otimes y_i'' \cdot x_1) + y_2 \otimes x_1 + x_1 \otimes y_2)$$

So a simple calculation proves the assertion.

One has

$$\pi_3(l_3(x)) = l_3(x) - \frac{1}{2}\mu(l_2(x) \otimes x) + \frac{1}{12}\mu(l_1(x) \otimes x \otimes x) \qquad (20)$$

where $l_i(x)$ is the generator of A_n corresponding to the classical n-logarithm. For $n = 2$ and $n = 3$ it is given by the picture on fig.1. Set

$$L_3(x) := Li_3(x) - \frac{1}{2}Li_2(x)\log x + \frac{1}{12}Li_1(x)(\log x)^2$$

Theorem 3.9

$$\Lambda_3(\pi_3(L, M)) = L_3(a_3(L, M)) \qquad (21)$$

Remark. To get a local coincidence of two multivalued analytic functions in this formula we should choose appropriate cycles of integration. The theorem claims in particular that it is possible to do this.

Proof. The main result of this paper implies that

$$\nu(\pi_3(L, M)) = \nu(l_3 \circ a_3(L, M))$$

Since the differential of $\Lambda_3(L, M)$ is determined by the coproduct of (L, M) (see the lemma below) this implies that the differentials of both sides of (21) coincide. So the difference between the left and right hand sides is a constant. Considering the additivity relation in L (which has odd number of terms) we deduce that this constant is zero.

Lemma 3.10 *Let $(L, M) \in A_n(\mathbb{C})$ and the $A_{n-1} \otimes A_1$ component of $\nu(L, M)$ is $\sum x_i \otimes y_i$. Then*

$$d\Lambda_3(L, M) = \sum d\Lambda_3(x_i) d \log y_i$$

This is a particular case of a general fact about the differential of the period of n-framed mixed Tate motives, see the chapter "Periods" in [G9]. A direct proof can be given by an explicit calculation using proposition 2.3.

4 Configurations of $2n$ points in $\mathbb{P}^{n-1}$, Grassmannian n-logarithms and motivic Lie coalgebra of a field

1. Grassmannian polylogarithms. Let me recall the construction of the Grassmannian polylogarithm function $\mathcal{L}_n^G(h_1, ..., h_{2n})$ given in [G4]. It is a function on configurations of *arbitrary* $2n$ hyperplanes in $\mathbb{CP}^{n-1}$.

Let $f_1, ..., f_m$ be m complex-valued functions on a manifold X. We attach to them the following $(m-1)$-form. Let $c_{j,m} := \frac{1}{(2j+1)!(m-2j-1)!}$. Set

$$\omega_{m-1}(f_1, ..., f_m) := \tag{22}$$

$$\frac{1}{(2\pi i)^m} \mathrm{Alt}_m \sum_{j \geq 0} c_{j,m} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2j+1}| \wedge di \arg f_{2j+2} \wedge ... \wedge di \arg f_m$$

Now let f_i be a rational function on $\mathbb{CP}^{n-1}$ with the divisor $(f_i) = h_i - h_{2n}$, $i = 1, ..., 2n-1$. Then

$$\mathcal{L}_n^G(h_1, ..., h_{2n}) := \int_{\mathbb{CP}^{n-1}} \omega_{2n-2}(f_1, ..., f_{2n-1})$$

Theorem 4.1 *$\mathcal{L}_n^G(h_1, ..., h_{2n})$ satisfies the following functional equations:*

a) For any $2n+1$ hyperplanes $h_1, ..., h_{2n+1}$ in $\mathbb{CP}^{n-1}$ one has

$$\sum_{i=1}^{2n+1} (-1)^i \mathcal{L}_n^G(h_1, ..., \hat{h}_i, ..., h_{2n+1}) = 0$$

b) For any $2n+1$ hyperplanes $p_1, ..., p_{2n+1}$ in $\mathbb{CP}^n$ one has

$$\sum_{i=1}^{2n+1} (-1)^i \mathcal{L}_n^G(p_1 \cap p_i, ..., p_{2n+1} \cap p_i) = 0$$

c) The function $\mathcal{L}_n^G(h_1, ..., h_{2n})$ is skewsymmetric with respect to $h_1, ..., h_{2n}$.

d) $\mathcal{L}_n^G(h_1, ..., h_{2n}) = 0$ if the intersection of certain $2k$ hyperplanes among $h_1, ..., h_{2n}$ has codimension $\leq k$.

Proof. a), b) are proved in [G4] and c) is clear.

d). One can show that up to a (computable) rational number d_n one has

$$\int_{\mathbb{CP}^{n-1}} \omega_{2n-2}(f_1, ..., f_{2n-1}) = d_n \cdot \int_{\mathbb{CP}^{n-1}} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2n-1}| \quad (23)$$

If $h_2 \cap ... \cap h_{2k+1}$ has codimension k then $d\log|f_2| \wedge ... \wedge d\log|f_{2k+1}| = 0$, so the integral is zero.

On the other hand since for $(L, M) \in A_n(\mathbb{C})$ the function $\Lambda_n(L, M)$ is a period of the mixed Hodge structure $H^n(\mathbb{CP}^n \backslash L, M)$, one can define a single valued function $\mathcal{L}_n(L, M) \in \mathbb{R}$ as the $\mathbb{R}$-period of this mixed Hodge structure. Then $\mathcal{L}_n : A_n(\mathbb{C}) \to \mathbb{R}$ is a homomorphism of groups.

Problems. 1. Find a complete list of functional equations for the function $\mathcal{L}_n^G$.

2. Find the relationship between the Aomoto and Grassmannian polylogarithms.

In the next section we will formulate these problems more precisely and explain their importance.

2. Conjectures. I conjecture that one can describe *explicitly* a certain subgroup

$$\mathcal{R}_n^G(F) \subset \bar{C}_{2n}(F)$$

such that $\mathcal{R}_n^G(\mathbb{C})$ is the group of all functional equations for the Grassmannian n-logarithm function $\mathcal{L}_n^G$. The subgroup should be given by an explicit "universally defined" finite list of "relations". More precisely, one should have a finite set of varieties $R_i(n)$ over $\mathbb{Z}$ and morphisms $s_i(n) : \mathbb{Z}[R_i(n)] \to \mathbb{Z}[\overline{C}_{2n}(\mathbb{P}^{n-1})]$ given by finite correspondences over $\mathbb{Z}$ such that

$$\mathcal{R}_n^G(F) = \sum s_i(n)\Big(\mathbb{Z}[R_i(n)]\Big)$$

The properties of the subgroups $\mathcal{R}_n^G(F)$ are formulated in the conjecture below. Then

$$G_n(F) = \quad \frac{\mathbb{Q}[\text{ configurations of any } 2n\ F\text{-points in } \mathbb{P}^{n-1}]}{\mathcal{R}_n^G(F)}$$

Then $\mathcal{L}_n^G : G_n(\mathbb{C}) \to \mathbb{R}, \quad (l_1, ..., l_{2n}) \longmapsto \mathcal{L}_n^G(l_1, ..., l_{2n})$.

Let $G_\bullet := \oplus_{n=1}^\infty G_n$. Recall that a structure of a Lie coalgebra on $G_\bullet$ is given by homomorphisms

$$G_n \xrightarrow{\delta_n} \oplus_{i \le n/2} G_i \wedge G_{n-i}$$

such that if $\delta := \oplus \delta_n$ then

$$G_\bullet \xrightarrow{\delta} \Lambda^2 G_\bullet \xrightarrow{\delta \otimes id - id \otimes \delta} \Lambda^3 G_\bullet \longrightarrow ...$$

is a complex.

Conjecture 4.2 *a) $G_\bullet$ has a natural structure of a graded Lie coalgebra over $\mathbb{Q}$.*

b) The category of graded finite dimensional modules over $G_\bullet(F)$ is equivalent to the category of mixed Tate motives over SpecF. In particular one should have

$$H^i_{(n)}\Big(G_\bullet(F)\Big) = gr^\gamma_n K_{2n-i}(F) \otimes \mathbb{Q}$$

Here $H^i_{(n)}$ is the degree n part of H^i. Set

$$G_1(F) := F^* \otimes \mathbb{Q}, \quad G_2(F) := B_2(F) \otimes \mathbb{Q}, \quad G_3(F) := B_3(F) \otimes \mathbb{Q} \tag{24}$$

Then the first components of the cobracket δ are given by

$$\delta_2 : B_2(F) \to \Lambda^2 F^*_{\mathbb{Q}}, \qquad \delta_3 : B_3(F) \to G_2(F) \otimes F^*_{\mathbb{Q}}$$

where the homomorphisms δ_2 and δ_3 were defined in s.3.1 and 3.2.

The 7-term relation for the generalized cross-ratio r_3 is rather mysterious. Its analog needed to define the group $G_4(F)$ is unknown. However the results of this paper suggest the following strategy. Denote by $\tilde{A}_n$ the free abelian group generated by the generators of the group A_n, i.e. by admissible pairs of simplices. Let us assume that we have defined already the subgroups $\mathcal{R}^G_m$ for $m < n$.

Conjecture 4.3 *There exists a homomorphism $a_n : \tilde{A}_n \to \overline{C}_{2n}$ such that*

a) The following diagram is commutative

$$\begin{array}{ccc} \tilde{A}_n & \stackrel{\nu}{\longrightarrow} & \oplus_{1\leq i\leq n} A_i \otimes A_{n-i} \\ \downarrow a_n & & \downarrow a_i \wedge a_{n-i} \\ \overline{C}_{2n} & \stackrel{\delta_n}{\longrightarrow} & \oplus_{i\leq n/2} G_i(F) \wedge G_{n-i}(F) \end{array}$$

b) $\mathcal{L}_n(L;M) = c_n \mathcal{L}^G_n(a_n(L,M))$ for any $(L;M) \in A_n(\mathbb{C})$, where c_n is a normalization constant.

(See also conjecture 1.42 in [G1]).

Assuming this we *introduce* $\mathcal{R}^G_n(F)$ as the image under the map a_n of the defining relations for the group A_n. Thus $\delta_n(\mathcal{R}^G_n) = 0$ and we are getting a commutative diagram

$$\begin{array}{ccc} A_n & \stackrel{\nu}{\longrightarrow} & \oplus_{1\leq i\leq n} A_i \otimes A_{n-i} \\ \downarrow a_n & & \downarrow a_i \wedge a_{n-i} \\ G_n(F) & \stackrel{\delta_n}{\longrightarrow} & \oplus_{i\leq n/2} G_i(F) \wedge G_{n-i}(F) \end{array}$$

The results of s. 3 show how this program works for G_3.

Recall the subgroup $P_n \in A_n$ defined in the section 3.6

Conjecture 4.4 *The map a_n induces an isomorphism*

$$\bar{a}_n : A_n/P_n \to G_n(F)$$

For $n = 2$ this was proved in [BGSV1-2].

For $n = 3$ one can prove that $a_3(P_3) = 0$. To show that $\bar{a}_3$ is an isomorphism one should construct a homomorphism $L_3 : B_3(F) \to A_3(F)$ splitting the map a_3. The homomorphism $L_3 : \mathbb{Z}[F^*] \to A_n, \quad \{x\}_3 \longmapsto L_3\{x\}_3$ is given by the right hand side of formula (20). Then one should prove that the map $L_3 : \mathbb{Z}[F^*] \to A_3/P_3$ is surjective and $L_3(R_3) = 0$.

Conjecture (4.4) just means that the dual to the Hopf algebra $A_\bullet := \oplus A_n$ is isomorphic to the universal enveloping algebra of the Lie algebra $G_\bullet^\vee$. (Here $A \to A^\vee$ is the duality between the ind and pro $\mathbb{Q}$-vector spaces). It seems quite remarkable that the universal enveloping algebra of $G_\bullet^\vee$ admits a completely different description (the only similar situation which comes to mind is Lusztig's construction of $U(\mathcal{N})$). So we get two different descriptions of the motivic Lie algebra (reflecting the properties of the Aomoto and Grassmannian polylogarithms). It is even more interesting that there are two more ways of thinking about the same Lie algebra (!): the wonderful "cycle" construction of Bloch and Kriz [BK] (so far the only one which is completely done), and the construction reflecting the properties of multiple polylogarithms ([G5]). This definitely shows the richness of the subject.

All constructions of different models of the motivic Hopf algebra of the category of mixed Tate motives are based on the following idea. The set of appropriately defined equivalence classes of n-framed mixed Tate motives over F form an abelian group $\mathcal{A}_n$ and $\mathcal{A}_\bullet := \oplus_{n \geq 1} \mathcal{A}_n$ is a commutative Hopf algebra. It is isomorphic to the fundamental Hopf algebra of the category of mixed Tate motives over F (see [BGSV1-2], [BMS] for the definition of n-framed mixed motives). Consider a universal variation of n-framed mixed Tate motives over a base X_n. For any F-point of $x \in X_n$ we get an element $m_n(x) \subset \mathcal{A}_n$. Universality of the variation means that the map $\mathbb{Z}[X_n(F)] \to \mathcal{A}_n$ is surjective. The kernel of this map is supposed to be described explicitly. So

$$\oplus_{n>0} \frac{\mathbb{Z}[X_n(F)]}{\mathrm{Ker} \quad m_n}$$

should have a natural structure of a Hopf algebra, and one needs to determine it. (The variation of mixed Tate motives for the "cycle" Hopf algebra of [BK] can be found in the last section of [G4]). So it is quite interesting that we could get a construction of a co-Lie algebra directly, without constructing first its universal enveloping algebra.

3. A canonical map $\mathbb{Z}[F^*] \to A_n(F)$. Let A be a commutative graded Hopf algebra with a product μ and coproduct ν, A_+ the kernel of the augmentation homomorphism and

$$\tilde{\nu} := \nu - (id \otimes 1 + 1 \otimes id) : A \longmapsto A_+^{\otimes 2}$$

the restricted coproduct. We define a map of graded vector spaces $\tilde{\nu}_{[k]} : A \longrightarrow A_+^{\otimes k}$ as a composition

$$A \xrightarrow{\tilde{\nu}} A_+ \otimes A_+ \xrightarrow{\tilde{\nu}\otimes id} A_+ \otimes A_+ \otimes A_+ \xrightarrow{\tilde{\nu}\otimes id} ... \xrightarrow{\tilde{\nu}\otimes id} A_+^{\otimes k}$$

Let $\mu_k : A^{\otimes k} \to A$ be the product map. Set

$$\pi := \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \mu_k \circ \tilde{\nu}_{[k]} = Id - \frac{1}{2}\mu \circ \tilde{\nu} + ...$$

(The map π_3 from the section 3.6 is a particular case of this)

Proposition 4.5 *$\pi^2 = \pi$ and $Ker\pi = P$.*

Proof. Since $\pi = id$ (mod P) one needs to show only that $P \subset Ker\pi$.

Now return to the Hopf algebra $A_\bullet(F)$. Let B_k be the Bernoulli numbers. Recall that $l_n(x)$ is an element of $A_n(F)$ corresponding to the classical n-logarithm.

Proposition 4.6

$$\pi(l_n(x)) = \sum_{k\geq 0} \frac{B_k}{k!} l_{n-k}(x) \cdot x^k$$

Here $x^m := x \cdot ... \cdot x$ (m times) is the product in $A_\bullet(F)$ of the element of A_1 corresponding to x under the canonical isomorphism $A_1(F) \to F^*$. Notice that this formula coincides with the formula for the function $\Lambda_n(z)$ in s. 4.1 of [BD]. However we get it in a quite different way.

Proof. Let

$$l(x,t) := \sum_{k\geq 1} l_k(x) t^k$$

Lemma 4.7 $\tilde{\nu} : l(x,t) \longmapsto l(x,t) \otimes (e^{x\cdot t} - 1)$

This is the generating function for the standard formula

$$\tilde{\nu}(l_k(x)) = \sum_{1\leq i\leq k-1} l_{k-i}(x) \otimes \frac{x^i}{i!}$$

for the coproduct of the classical polylogarithm. Therefore

$$\mu \circ \nu_{[k]} : \quad l(x,t) \longmapsto l(x,t) \cdot (e^{x\cdot t} - 1)^{k-1}$$

So

$$\pi : l(x,t) \longmapsto \quad -\sum_{k\geq 1} \frac{(-1)^k}{k} l(x,t) \cdot (e^{x\cdot t} - 1)^{k-1} =$$

$$= l(x,t)\sum_{k\geq 1}\frac{(1-e^{x\cdot t})^k}{k}\frac{1}{e^{x\cdot t}-1} \quad = \quad l(x,t)\cdot\frac{xt}{e^{x\cdot t}-1} \quad = \quad \sum_{n\geq 1}\Big(\sum_{k\geq 0}\frac{B_k}{k!}l_{n-k}(x)\cdot x^k\Big)t^n$$

The proposition is proved.

The kernel of the map

$$L_n : \mathbb{Z}[F^*] \longmapsto A_n(F), \quad \{x\} \longmapsto \pi(l_n(x))$$

should coincide with the subgroup of all functional equations for the n-logarithm.

5 Motivic structure of the Grassmannian tetralogarithm and Lie coalgebra $G(F)_{\leq 4}$

1. The group $\tilde{G}_{2n}(F)$. Let $\tilde{G}_{2n}(F)$ be the free abelian group generated by 2n-tuples of points $(l_1,...,l_{2n})$ in generic position in $\mathbb{P}^{n-1}(F)$ subject to the following relations:

1)*Projective invariance*: $(l_1,...,l_{2n}) = (gl_1,...,gl_{2n})$ for any $g \in PGL_n(F)$.

2) *Skew symmetry.* $(l_1,...,l_{2n}) = (-1)^{|\sigma|}(l_{\sigma(1)},...,l_{\sigma(2n)})$ for any $\sigma \in S_{2n}$.

3) $(2n+1)$ *-term relation*: for any $2n+1$ points in generic position $(l_0,...,l_{2n})$ in $P^{n-1}(F)$ one has $\sum_{i=0}^{2n}(-1)^i(l_0,...\hat{l}_i,...,,l_{2n}) = 0$

4)*dual* $(2n+1)$ *-term relation*: for any $2n+1$ points in generic position $(l_0,...,l_{2n})$ in $\mathbb{P}^n(F)$ one has $\sum_{i=0}^{2n}(-1)^i(l_i|l_0,...\hat{l}_i,...,,l_{2n}) = 0$

Remark. These relations reflect the properties of the Grassmannian n-logarithm listed in theorem 4.1. The group $G_{2n}(F)$ is supposed to be a quotient of the group $\tilde{G}_{2n}(F)$. It is a nontrivial quotient already for $n = 3$.

2. The main result. After identification (24) the degree 4 part of the cochain complex of the Lie coalgebra $G_\bullet(F)$ should look as follows:

$$G_4(F) \quad \xrightarrow{\delta} \quad B_3(F)\otimes F^* \quad \oplus \quad B_2(F)\wedge B_2(F) \xrightarrow{\delta} B_2(F)\otimes\Lambda^2F^* \xrightarrow{\delta} \Lambda^4F^*$$

Here

$$\delta : \{x\}_3 \otimes y \longmapsto \{x\}_2 \otimes x \wedge y,$$

$$\delta : \{x\}_2 \wedge \{y\}_2 \longmapsto \{y\}_2 \otimes (1-x)\wedge x - \{x\}_2 \otimes (1-y)\wedge y$$

It remains to define a homomorphism

$$\tilde{G}_4(F) \quad \xrightarrow{\delta} \quad B_3(F)\otimes F^* \quad \oplus \quad B_2(F)\wedge B_2(F)$$

Below we will construct it as a composition

$$\tilde{G}_4(F) \quad \xrightarrow{\delta} \quad A_3^0(F)\otimes F^* \quad \oplus \quad F^*\otimes A_3^0(F) \quad \oplus \quad B_2(F)\wedge B_2(F)$$

$$\xrightarrow{(a_3\otimes id - id\otimes a_3, Id)} B_3(F)\otimes F^* \quad \oplus \quad B_2(F)\wedge B_2(F)$$

Namely, $\delta = (\delta_{3,1}, \delta_{1,3}, \delta_{2,2})$ where

$$\delta_{3,1}(l_1, ..., l_8) := \mathrm{Alt}_8\Big((l_1, l_2, l_3, l_4; l_5, l_6, l_7, l_8) \otimes \Delta(l_5, l_6, l_7, l_8)\Big) \in A_3^0(F) \otimes F^*$$

$$\delta_{1,3}(l_1, ..., l_8) := -\mathrm{Alt}_8\Big(\Delta(l_1, l_2, l_3, l_4) \otimes (l_1, l_2, l_3, l_4; l_5, l_6, l_7, l_8)\Big) \in F^* \otimes A_3^0(F)$$

$$\delta_{2,2}(l_1, ..., l_8) := \frac{288}{7} \cdot \mathrm{Alt}_8\Big((l_1, l_2|l_3, l_4, l_5, l_6)_2 \wedge (l_3, l_4|l_1, l_2, l_5, l_7)_2\Big) \in \Lambda^2 B_2(F)$$

Theorem 5.1 *a) The homomorphism δ is well defined, i.e. sends the relations 1) - 4) to zero.*

b) The composition

$$\tilde{G}_4(F) \stackrel{\delta}{\longrightarrow} B_3(F) \otimes F^* \quad \oplus \quad B_2(F) \wedge B_2(F) \stackrel{\delta}{\longrightarrow} B_2(F) \otimes \Lambda^2 F^*$$

is zero.

The proof is postponed to Section 5.6.

Remark. Taking, as usual ([G1]), the "connected component" of zero in $\mathrm{Ker}\delta$ we should get the set of defining relations for the group $G_4(F)$. However an explicit construction of them is not known yet.

3. Applications to the Borel regulator map. Recall the rank filtration quotient $K_7^{[3]}(F) \otimes \mathbb{Q}$ of $K_7(F) \otimes \mathbb{Q}$, which is expected to be isomorphic to $gr_4^{\gamma} K_7(F) \otimes \mathbb{Q}$, see [G1].

It turn out that using the results of section 3.3, namely the definition of the map a_3 given there and theorem 3.3, we can factorize the natural projection of the map δ to $B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)$, i.e. the map

$$C_8(V_4) \longrightarrow B_2(F) \otimes \Lambda^2 F^*$$

as a composition

$$C_8(V_4) \stackrel{\partial}{\longrightarrow} C_7(V_4) \stackrel{f_7(4)}{\longrightarrow} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)$$

where the map

$$f_7(4) : C_7(V_4) \longrightarrow B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)$$

is defined as follows.

We will use shorthands like :

$$(1, 2, 3, 4) \quad \text{for} \quad \Delta(l_1, l_2, l_3, l_4) \quad \text{and so on}$$

Write $f_7(4) = f_{2,2} + f'_{3,1} + f''_{3,1}$, where

$$f_{2,2} : (l_1, ..., l_7) \longmapsto \frac{288}{7} \cdot \mathrm{Alt}_7\Big((l_1, l_2|l_3, l_4, l_5, l_6)_2 \wedge (l_3, l_4|l_1, l_2, l_5, l_7)_2\Big) \in \Lambda^2 B_2(F)$$

$$f'_{3,1} : (l_1, ..., l_7) \longmapsto \quad \frac{-32}{6} \cdot \mathrm{Alt}_7 \Big((l_1 | l_2, l_3, l_4, l_5, l_6, l_7)_3 \otimes \Delta(l_1, l_2, l_3, l_4) \Big) \in B_3(F) \otimes F^*$$

$$f''_{3,1} : (l_1, ..., l_7) \longmapsto \quad 96 \cdot \mathrm{Alt}_7 \Big(\{ r(l_1, l_5 | l_2, l_6, l_3, l_7) \}_3 \otimes \Delta(l_1, l_2, l_3, l_4) \Big) \in B_3(F) \otimes F^*$$

Notice that if we divide all the coefficients by 32 the coefficients will be smaller: $\frac{9}{7}, \frac{-1}{6}, 3$. We will use a notation $f_{3,1}$ for $f'_{3,1} + f''_{3,1}$. Finally, $(l_1 | l_2, l_3, l_4, l_5, l_6, l_7)_3 \in B_3(F)$ is obtained as follows: the configuration $(l_1 | l_2, l_3, l_4, l_5, l_6, l_7)$ of 6 points in P^2 provides a generator of $G_3(F)$, which is then mapped to $B_3(F)$ using the generalized cross-ratio r_3.

To get the formula for $f''_{3,1}$ we have used the definition of a''_3 and the intermediate formula

$$f''_{3,1} \circ \partial : (l_1, ..., l_8) \longmapsto \quad \frac{-32}{3} \cdot \mathrm{Alt}_8 \Big(\mu_3(l_1, l_5 | l_2, l_3, l_4; l_6, l_7, l_8) \otimes \Delta(l_1, l_2, l_3, l_4) \Big)$$

Therefore there is the following commutative diagram:

$$\begin{array}{ccccccc}
C_9(V_4) & \xrightarrow{\partial} & C_8(V_4) & \xrightarrow{\partial} & C_7(V_4) & & \\
\downarrow & & \downarrow f_8(4) & & \downarrow f_7(4) & & \\
0 & \longrightarrow & \tilde{G}_4(F) & \xrightarrow{\delta} & B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F) & \longrightarrow & B_2(F) \otimes \Lambda^2 F^*
\end{array}$$

Moreover, it is easy to see that the composition

$$C_9(V_5) \xrightarrow{\partial'} C_8(V_4) \longrightarrow B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)$$

is zero. Here $(l_1, ..., l_9) \longmapsto \sum (-1)^i (l_i | l_1, ..., \widehat{l_i}, ..., l_9)$.

Therefore we have constructed a morphism from the appropriate part of the weight 4 bigrassmannian complex ([G6]) to the bottom line in the diagram above. (The full homomorphism from the weight 4 bigrassmannian complex to the weight 4 motivic complex will be treated in [G8]). Applying the general technique developed in [G1-2] and [G6] we get part a) of the following theorem.

Theorem 5.2 *a) There exists a canonical map*

$$K_7^{[3]}(F) \otimes \mathbb{Q} \quad \longrightarrow \quad \mathrm{Ker} \Big(\tilde{G}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F) \Big)_{\mathbb{Q}}$$

b) In the case $F = \mathbb{C}$ the composition

$$K_7^{[3]}(\mathbb{C}) \longrightarrow \tilde{G}_4(\mathbb{C}) \xrightarrow{\mathcal{L}_4^G} \mathbb{R}$$

coincides with a nonzero rational multiple of the Borel regulator map.

Namely, the diagram above provides a morphism from a piece of the weight four bigrassmannian complex to the complex

$$0 \longrightarrow \tilde{G}_4(F)_{\mathbb{Q}} \xrightarrow{\delta} (B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F))_{\mathbb{Q}}$$

Combining this with the canonical maps from the homology of $GL(F)$ to the weight four complex of affine flags ([G6]) followed by the canonical map from the complex of affine flags to the weight four bigrassmannian complex we get a canonical map

$$H_7(GL(F), \mathbb{Q}) \quad \longrightarrow \quad \mathrm{Ker}\Big(\tilde{G}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)\Big)_{\mathbb{Q}}$$

Using the arguments given in the papers cited above we get part a).

To get part b) we use the computation of the Borel regulator map via the Grassmannian n-logarithm $\mathcal{L}_n^G$ (see [G4] and [G7]).

Conjecture 5.3 *There exists a map*

$$\mathrm{Ker}\Big(\tilde{G}_4(F) \xrightarrow{\delta} B_3(F) \otimes F^* \oplus \Lambda^2 B_2(F)\Big)_{\mathbb{Q}} \quad \longrightarrow \quad K_7^{[3]}(F) \otimes \mathbb{Q}$$

which in the case $F = \mathbb{C}$ commutes with the Borel regulator map.

4. Towards Zagier's conjecture on $\zeta_F(4)$. Let us construct a homomorphism

$$\overline{f}_7(4) : C_7(4) \longrightarrow B_3(F) \otimes F^* \qquad (25)$$

providing a definition of a homomorphism $\overline{\delta} : G_4(F) \longrightarrow B_3(F) \otimes F^*$ whose composition with the natural map $B_3(F) \otimes F^* \longrightarrow B_2(F) \otimes \Lambda^2 F^*$ is zero. So we will get a commutative diagram

$$\begin{tikzcd} C_8(4) \arrow[r, "\partial"] \arrow[d, "f_8(4)"'] & C_7(4) \arrow[d, "\overline{f}_7(4)"] \\ G_4(F) \arrow[r, "\overline{\delta}"] & B_3 \otimes F^* \end{tikzcd}$$

This is done in 2 steps. First we use the formulas for the map $f_7(4)$ given in s. 5.3 as a definition of a homomorphism

$$C_7(V_4) \overset{f_{3,1} \oplus f_{2,2}}{\longrightarrow} B_3 \otimes F^* \oplus \Lambda^2 \mathbb{Z}[P_F^1] \qquad (26)$$

Then we construct a homomorphism

$$g : \Lambda^2 \mathbb{Z}[P_F^1] \longrightarrow B_3 \otimes F^*$$

making the diagram

$$
\begin{array}{ccc}
\Lambda^2\mathbb{Z}[P^1_F] & & \\
 & \searrow\tilde{\delta} & \\
\downarrow g & & B_2\otimes\Lambda^2F^* \\
 & \nearrow\delta & \\
B_3\otimes F^* & &
\end{array}
$$

commutative, and so providing a commutative diagram

$$
\begin{array}{ccc}
B_3\otimes F^*\oplus\Lambda^2\mathbb{Z}[P^1_F] & \xrightarrow{\delta+\tilde{\delta}} & B_2\otimes\Lambda^2F^* \\
\Big\downarrow id\oplus g & & \Big\downarrow id \\
B_3\otimes F^* & \xrightarrow{\delta} & B_2\otimes\Lambda^2F^*
\end{array}
\qquad *
$$

Here

$$\tilde{\delta}\Big(\{x\}\wedge\{y\}\Big)=\{y\}_2\otimes(1-x)\wedge x-\{x\}_2\otimes(1-y)\wedge y$$

To define g recall that the group S_3 acts naturally on $P^1\backslash\{0,1,\infty\}$. The orbit of a point x is $x,\quad x^{-1},\quad 1-x,\quad (1-x)^{-1},\quad 1-x^{-1},\quad (1-x^{-1})^{-1}$. Set

$$g:\quad \{x\}_2\wedge\{y\}_2\longrightarrow-\frac{1}{12}\sum_{\sigma_1,\sigma_2\in S_3}(-1)^{|\sigma_1|+|\sigma_2|}\{\frac{\sigma_1(x)}{\sigma_2(y)}\}_3\otimes\frac{1-\sigma_1(x)}{1-\sigma_2(y)}\tag{27}$$

Notice that $\{x^{-1}\}_3=\{x\}_3$ and so the right-hand side is skew-symmetric with respect to transposition of x and y.

We define the desired homomorphism (25) as the composition of morphismes (26) and $(\mathrm{id}\oplus g)$.

Remark. g does not factorizes through a homomorphism of $\Lambda^2B_2(F)$. In fact we proved in section 4 of [G2] that there is no "natural" (i.e. given by formulas) non-zero homomorphism $\Lambda^2B_2(F)\longrightarrow B_3(F)\otimes F^*$.

Proposition 5.4 *One has $\delta\circ g=\tilde{\delta}$. Therefore the diagram $(*)$ is commutative.*

Proof. Computing the coboundary δ of expression (27) we get

$$\frac{1}{12}\sum_{\sigma_1,\sigma_2\in S_3}(-1)^{|\sigma_1|+|\sigma_2|}\{\frac{\sigma_1(x)}{\sigma_2(y)}\}_2\otimes\frac{\sigma_1(x)}{\sigma_2(y)}\wedge\frac{1-\sigma_1(x)}{1-\sigma_2(y)}\in B_2(F)\otimes\Lambda^2F^*\tag{28}$$

$\sigma_1(x)\wedge(1-\sigma_1(x))\in\Lambda^2F^*$ is skewsymmetric with respect to S_3. So $(1-x)\wedge x$ appears in (28) with factor

$$-\frac{1}{12}\sum_{\sigma_1,\sigma_2\in S_3}(-1)^{|\sigma_2|}\{\frac{\sigma_1(x)}{\sigma_2(y)}\}_2\tag{29}$$

Modulo 2-torsion one can rewrite the basic relation in $B_2(F)$ as follows

$$\{x\}_2 - \{y\}_2 = \{\frac{x}{y}\} - \{\frac{1-x}{1-y}\} + \{\frac{1-x^{-1}}{1-y^{-1}}\} \tag{30}$$

Averaging with respect to x over the group S_3 we get

$$-6\{y\}_2 = \sum_{\sigma_1 \in S_3} \{\frac{\sigma_1(x)}{y}\}_2 - \{\frac{\sigma_1(x)}{1-y}\}_2 + \{\frac{\sigma_1(x)}{1-y^{-1}}\}_2 \tag{31}$$

So modulo 6-torsion (29) coincides with $\frac{1}{2}(\{y\}_2 - \{\frac{y}{y-1}\}_2) = \{y\}_2$. The proposition is proved.

A definition of the homomorphism $\overline{\delta} : G_4(F) \longrightarrow B_3(F) \otimes F^*$. Take any 8 points $(l_1, ..., l_8)$ in generic position in $\mathbb{P}^3(F)$. Lift them to 8 vectors $(\tilde{l}_1, ..., \tilde{l}_8)$ in the 4-dimensional vector space V_4 and then apply to them the homomorphism $\overline{f}_7(4) \circ \partial$. We claim that the result does not depend on the choise of vectors $\tilde{l}_i$ projecting to the points l_i. Indeed, only $f_{3,1}$ component of the map $\overline{f}_7(4)$ may depend on this choise, and it is easy to check it does not.

Recall the group $\mathcal{B}_4(F)$ defined in [G1], [G2].

Conjecture 5.5 *There exists a canonical homomorphism of groups* $\mathbb{L}_4 : G_4(F) \longrightarrow \mathcal{B}_4(F)$ *making the following diagram commutative:*

$$\begin{array}{ccc} G_4(F) & \xrightarrow{\overline{\delta}} & B_3(F) \otimes F^* \\ \mathbb{L}_4 \downarrow & & \downarrow = \\ \mathcal{B}_4(F) & \xrightarrow{\delta} & B_3(F) \otimes F^* \end{array}$$

It follows from theorem 5.2 that this conjecture implies Zagier's conjecture for $\zeta_F(4)$ for any number filed F.

5. A motivic construction of the Grassmannian tetralogarithm. Theorem 5.1 allows us to construct a tetralogarithm function $\tilde{\mathcal{L}}_4^G$ on configurations of 8 points in $\mathbb{CP}^3$ providing a homomorphism $\tilde{G}_4(\mathbb{C}) \longrightarrow \mathbb{R}$. Namely, let X be a variety over $\mathbb{C}$ and $F := \mathbb{C}(X)$. Then there is a homomorphism of complexes

$$\begin{array}{ccccc} B_3(F) \otimes F^* \quad \oplus \quad B_2(F) \wedge B_2(F) & \xrightarrow{\delta} & B_2(F) \otimes \Lambda^2 F^* & \xrightarrow{\delta} & \Lambda^4 F^* \\ \downarrow r_4(2) & & \downarrow r_4(3) & & \downarrow r_4(4) \\ S^1(SpecF) & \xrightarrow{d} & S^2(SpecF) & \xrightarrow{d} & S^3(SpecF) \end{array}$$

where $S^\bullet(SpecF)$ is the de Rham complex of smooth forms at the generic point of X over $\mathbb{C}$, given by the following formulas. Set

$$\widehat{\mathcal{L}_n}(z) = \begin{cases} \mathcal{L}_n(z) & n: \text{ odd} \\ i\mathcal{L}_n(z) & n: \text{ even} \end{cases}$$

and

$$\alpha(g_1, g_2) := -\log|g_1| d\log|g_2| + \log|g_2| d\log|g_1|$$

We define homomorphisms $r_4(\bullet)$ by the formulas

$$r_4(2) : \{f\}_3 \otimes g \mapsto \widehat{\mathcal{L}}_3(f) di\arg g - \tfrac{1}{3}\widehat{\mathcal{L}}_2(f)\log|g| \cdot d\log|f|$$

$$r_4(2) : \{f\}_2 \wedge \{g\}_2 \mapsto \tfrac{1}{3} \cdot \Big(\widehat{\mathcal{L}}_2(g) \cdot \alpha(1-f, f) - \widehat{\mathcal{L}}_2(f) \cdot \alpha(1-g, g)\Big)$$

$$r_4(3) : \{f\}_2 \otimes g_1 \wedge g_2 \mapsto \widehat{\mathcal{L}}_2(f) di\arg g_1 \wedge di\arg g_2 - \tfrac{1}{3}\alpha(1-f, f)\wedge$$
$$\Big(\log|g_1| d\arg|g_2| - \log|g_2| d\arg|g_1|\Big) \quad + \tfrac{1}{3}\widehat{\mathcal{L}}_2(f) d\log|g_1| \wedge d\log|g_2|$$

$$r_4(4) : f_1 \wedge ... \wedge f_4 \longmapsto \omega_3(f_1 \wedge ... \wedge f_4)$$

A direct computation shows that we get a morphism of complexes.

It follows from the theorem that the composition $r_4(2) \circ \delta(l_1, ..., l_8)$ is a closed 1-form on the space of generic configurations of 8 points in $\mathbb{P}^3$. It turns out that integrating it we get a (single-valued) function $\tilde{\mathcal{L}}_4^G$:

Proposition 5.6 *The integral* $\int_\gamma r_4(2) \circ \delta(l_1, ..., l_8)$ *is a single-valued function defined on the space of configurations of* 8 *points in* $\mathbb{P}^3$ *in generic position.*

Here we integrate along a path γ from a given reference point to a variable point in the configuration space. The constant of integration is normalized by the condition that the function tends to zero when the configuration degenerates.

Proof. The fundamental group of the configuration space is generated by loops around divisors $\Delta(l_{i_1}, ..., l_{i_4}) = 0$. It is clear from the formula for $r_4(2)(\{f\}_2 \wedge \{g\}_2)$ that the $B_2 \wedge B_2$ part of the 1-form $r_4(2) \circ \delta(l_1, ..., l_8)$ has trivial monodromy. Further, the monodromy around $\Delta(l_1, ..., l_4) = 0$ is equal to the limit value of $2 \cdot 2\pi i \widehat{\mathcal{L}}_3 \circ a_3((l_1, l_2, l_3, l_4; l_5, l_6, l_7, l_8))$ at the divisor $\Delta(l_1, ..., l_4) = 0$, which is zero. The proposition is proved.

We have constructed two functions, $\tilde{\mathcal{L}}_4^G$ and $\mathcal{L}_4^G$, on configurations of 8 points in $\mathbb{P}^3$ satisfying the properties listed in theorem 4.1.

Question 5.7 *Does the function* $\tilde{\mathcal{L}}_4^G$ *coincides with a multiple of the Grassmannian tetralogarithm* $\mathcal{L}_4^G$ *constructed in section 3. More precisely,* $\tilde{\mathcal{L}}_4^G = \lambda(2\pi)^3 \cdot \mathcal{L}_4^G$ *where* $\lambda \in \mathbb{Q}^*$*?*

I am completely sure these two functions essentially coinside.

A version of conjecture 5.5. The composition

$$G_4(\mathbb{C}) \xrightarrow{\overline{\delta}} B_3(\mathbb{C}) \otimes \mathbb{C}^* \xrightarrow{r_4(2)} \quad \text{1-forms}$$

provide *another* 1-form on the configuration space of 8 points. It is closed by the main result of this section. Therefore we can integrate it and get a function, which turns out to be single-valued (the same argument as above), denoted $\overline{\mathcal{L}}_4^G$.

Conjecture 5.5 implies that this function is expressable via the classical 4-logarithm.

6. Proof of theorem 5.1. It consists of two different parts. We will first compute the image in $B_2 \otimes \Lambda^2 F^*$ of the $\delta_{1,3} + \delta_{3,1}$-component of δ, then do the same for the $\delta_{2,2}$-component, and will see that they differ by a sign.

Part 1. During the proof of this theorem we will use shorthands like :

$$(1,2|3,4,5,6)_2 \quad \text{for} \quad \{r(l_1,l_2|l_3,l_4,l_5,l_6)\}_2$$

Lemma 5.8 *The following composition*

$$\tilde{G}_4 \xrightarrow{\delta_{3,1}} A_3^0 \otimes F^* \xrightarrow{\nu_{1,2}\otimes \mathrm{id}} F^* \otimes A_2 \otimes F^* \xrightarrow{p} B_2 \otimes \Lambda^2 F^*,$$

where $p : x_1 \otimes y_2 \otimes z_1 \longmapsto -a_2(y_2) \otimes x_1 \wedge z_1$, *is equal to zero.*

Proof. Using proposition 2.3

$$p \circ (\nu_{1,2} \otimes \mathrm{id}) \circ \delta_{3,1} : \quad (l_1, ..., l_8) \longmapsto$$

$$-16 \cdot \mathrm{Alt}_8\Big(a_2(5|2,3,4;6,7,8) \otimes (5,2,3,4) \wedge (5,6,7,8)\Big) = 0$$

Indeed, the expression we alternate is symmetric with respect to the odd involution exchanging $(2,3,4)$ with $(6,7,8)$. Here we used the formula $a_2(L,M) = -a_2(M,L)$ ([BGSV]).

Let us compute the composition

$$G_4 \xrightarrow{\delta_{3,1}} A_3^0 \otimes F^* \xrightarrow{\nu_{2,1}\otimes \mathrm{id}} A_2 \otimes F^* \otimes F^* \longrightarrow B_2 \otimes \Lambda^2 F^* \tag{32}$$

Lemma 5.9 *The composition (32) is given by*

$$(l_1, ..., l_8) \longmapsto -144 \cdot \mathrm{Alt}_8\Big((3,4|1,2,6,7)_2 \otimes (1,2,3,4) \wedge (1,2,3,5)\Big)$$

Proof. Using proposition 2.3 we get

$$\nu_{2,1} \otimes \mathrm{id} \circ \delta_{3,1} : \quad (l_1, ..., l_8) \longmapsto$$

$$16 \cdot \mathrm{Alt}_8\Big((1|2,3,4;6,7,8) \otimes (1,6,7,8) \wedge (5,6,7,8)\Big)$$

Applying formula (6) for a_2 and using $a_2(L,M) = -a_2(M,L)$ we get

$$-144 \cdot \mathrm{Alt}_8\Big((1,6|7,8,3,4)_2 \otimes (1,6,7,8) \wedge (5,6,7,8)\Big)$$

Using the even permutation $(1,2,3,4,5,6,7,8) \longmapsto (4,8,6,7,5,3,1,2)$ we get the lemma.

The computations for $\delta_{1,3}$ are completely similar and can be formally deduced from the computations for $\delta_{3,1}$ using the following fact. Consider the composition

$$A_3^0 \quad \xrightarrow{\nu_3} \quad A_2 \otimes A_1 \quad \oplus \quad A_1 \otimes A_2 \quad \xrightarrow{a_2 \wedge a_1} \quad B_2 \otimes F^*$$

where $a_2 \wedge a_1$ was defined in s. 3.3. Then $a_2 \wedge a_1 \circ \nu_3(L, M) = a_2 \wedge a_1 \circ \nu_3(M, L)$.

Part 2. We will use a lot the five term relation $\sum(-1)^i(l_i|l_1, ..., \widehat{l_i}, ..., l_5)_2 = 0$.

Since

$$\delta(1, 2|3, 4, 5, 6)_2 = \frac{1}{2}\mathrm{Alt}_{\{3,4,5,6\}}((1, 2, 3, 4) \wedge (1, 2, 3, 5))$$

and

$$\mathrm{Alt}_8\Big((1, 2|3, 4, 5, 6)_2 \wedge (3, 4|1, 2, 5, 7)_2\Big) =$$

$$-\mathrm{Alt}_8\Big((3, 4|1, 2, 5, 7)_2 \wedge ((1, 2|3, 4, 5, 6)_2\Big)$$

one has

$$\delta \circ \mathrm{Alt}_8\Big(((1, 2|3, 4, 5, 6)_2 \wedge (3, 4|1, 2, 5, 7)_2\Big) =$$

$$= \mathrm{Alt}_8\Big((3, 4|1, 2, 5, 7)_2 \otimes \mathrm{Alt}_{\{3,4,5,6\}}[(1, 2, 3, 4) \wedge (1, 2, 3, 5)]\Big) =$$

$$\mathrm{Alt}_8\Big((3, 4|1, 2, 5, 7)_2 \otimes [-2 \cdot (1, 2, 3, 4) \wedge (1, 2, 3, 6) + 2 \cdot (1, 2, 3, 5) \wedge (1, 2, 3, 6)-$$

$$2 \cdot (1, 2, 3, 5) \wedge (1, 2, 5, 6) + 2 \cdot (1, 2, 3, 6) \wedge (1, 2, 5, 6) - (1, 2, 3, 6) \wedge (1, 2, 4, 6)]\Big)$$

In the last step we have used the following simple observation: the terms in the $\Lambda^2 F^*$-factor where 6 is absent vanish by skew-symmetry, since 6 and 8 are also absent in $(3, 4|1, 2, 5, 7)_2$.

Using the skew-symmetry we rewrite the last formula as follows:

$$\mathrm{Alt}_8\Big(\Big(2 \cdot (3, 4|1, 2, 6, 7)_2 + 2 \cdot (3, 6|1, 2, 5, 7)_2 + 2 \cdot (4, 6|1, 2, 3, 7)_2$$
$$+2 \cdot (4, 6|1, 2, 5, 7)_2 + (5, 4|1, 2, 6, 7)_2\Big) \otimes (1, 2, 3, 4) \wedge (1, 2, 3, 5)\Big) \tag{33}$$

Lemma 5.10 *The expression (33) is equal to*

$$7 \cdot \mathrm{Alt}_8\Big((3, 4|1, 2, 6, 7)_2 \otimes (1, 2, 3, 4) \wedge (1, 2, 3, 5)\Big)$$

Proof. Computing

$$\mathrm{Alt}_8\Big(2\cdot(3,6|1,2,5,7)_2\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)= \tag{34}$$

$$\mathrm{Alt}_8\Big([(3,6|1,2,5,7)_2-(3,7|1,2,5,6)_2]\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)$$

using the five term relation

$$(3,6|1,2,5,7)_2-(3,7|1,2,5,6)_2=(3,5|1,2,6,7)_2-(3,2|1,5,6,7)_2+(3,1|2,5,6,7)_2$$

we see that the contribution of each of the last two terms is zero because of the skewsymmetry in $(2,3)$ and $(1,3)$. So we get, using skew-symmetry in $(4,5)$,

$$(34)\quad=\quad\mathrm{Alt}_8\Big((3,4|1,2,6,7)_2\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)$$

A similar consideration using the five-term relation

$$(4,6|1,2,3,7)_2-(4,7|1,2,3,6)_2=(4,3|1,2,6,7)_2-(4,2|1,3,6,7)_2+(4,1|2,3,6,7)_2=0$$

and skew-symmetry in $(1,2,3)$ gives us

$$\mathrm{Alt}_8\Big([2\cdot(4,6|1,2,3,7)_2-3\cdot(3,4|1,2,6,7)_2]\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)=0$$

Using the five term relation

$$(4,6|1,2,5,7)_2-(4,6|1,3,5,6)_2+(4,6|2,3,5,7)_2-(4,6|1,2,3,7)_2+(4,6|1,2,3,5)_2=0$$

we get

$$\frac{1}{3}\cdot\mathrm{Alt}_8\Big(2\cdot(4,6|1,2,5,7)_2\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)= \tag{35}$$

$$\frac{1}{3}\cdot\mathrm{Alt}_8\Big([(4,6|1,2,3,7)_2-(4,6|1,2,3,5)_2]\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)$$

The last term gives zero contribution since it does not contain 7 and 8. Using the five term relation

$$[(4,6|1,2,3,7)_2-[(4,7|1,2,3,6)_2+[(4,1|2,3,6,7)_2-[(4,2|1,3,6,7)_2+[(4,3|1,2,6,7)_2$$

we conclude that multiplying (35) by 3 we get

$$\mathrm{Alt}_8\Big((3,4|1,2,6,7)_2\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)$$

Finally,

$$\mathrm{Alt}_8\Big((4,5|1,2,6,7)_2\otimes(1,2,3,4)\wedge(1,2,3,5)\Big)=0$$

for the following reason. Write it as

$$\frac{1}{3} \cdot \mathrm{Alt}_8 \circ \mathrm{Alt}_{\{1,2,3\}} \Big((4,5|1,2,6,7)_2 \otimes (1,2,3,4) \wedge (1,2,3,5) \Big)$$

Using the five term relation for the configuration $(4,5|1,2,3,6,7)$ we can write it as

$$\frac{1}{3} \cdot \mathrm{Alt}_8 \circ \mathrm{Alt}_{\{6,7\}} \Big((4,5|1,2,3,6)_2 \otimes (1,2,3,4) \wedge (1,2,3,5) \Big)$$

Each of the two terms (before taking Alt_8) in this expression is zero since the first one does not contain $(7,8)$ and the second $(6,8)$.

The lemma is proved.

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Dept of Mathematics, Brown University, Providence RI 02912, USA. e-mail: sasha@math.brown.edu